Do three of the problems from section A and three questions from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have counted. If you have doubts about the wording of a problem or about what results may be assumed without proof, please ask for clarification. In no case should you interpret a problem in such a way that it becomes trivial.

Section A:

1. Show that every compact metrizable space has a countable basis.

2. For a topological space X and $y \in X$, the *path component* P_y of X containing y is the largest path-connected subset with $y \in P_y \subseteq X$.
(a) Show that this concept is well-defined (that is, show that every point y is contained in a largest path-connected subset).
(b) Give an example of a space and point $y \in X$ so that P_y is neither an open nor a closed subset of X.

3. Let S^1 and D^2 be the unit circle and closed unit disk in the (Euclidean) space $\mathbb{R}^2$. Let $\sim$ be the smallest equivalence relation on the product space $S^1 \times D^2$ satisfying $(x, y) \sim (x, (0, 0))$ for all $x \in S^1$ and $y \in D^2$. Let $Z := (S^1 \times D^2)/\sim$ be the corresponding quotient space. Prove that Z is homeomorphic to S^1.

4. Let $(X, \mathcal{T})$ and $(X, \mathcal{T}')$ be topological spaces with $\mathcal{T} \subseteq \mathcal{T}'$.
(a) If $(X, \mathcal{T}')$ is normal, must $(X, \mathcal{T})$ also be normal?
(b) If $(X, \mathcal{T}')$ is compact, must $(X, \mathcal{T})$ also be compact?
(Prove your answers in both parts.)

Section B:

5. Let D^2 be the closed unit disk in $\mathbb{R}^2$ and let $\sim$ be the smallest equivalence relation on D^2 satisfying $(\cos(\theta), \sin(\theta)) \sim (\cos(\theta + \frac{2\pi}{3}), \sin(\theta + \frac{2\pi}{3}))$ and $(\cos(\theta), \sin(\theta)) \sim (\cos(-\theta), \sin(-\theta))$ for all $\theta \in [0, \frac{2\pi}{3}]$. Let $X := D^2/\sim$ be the corresponding quotient space.
(a) Compute a presentation for $\pi_1(X)$.
(b) Is X homotopy equivalent to S^1? (Prove your answer.)

6. Let $p : \widetilde{X} \to X$ be a covering space. Show that if X is Hausdorff, then $\widetilde{X}$ is also Hausdorff.

7. Use covering space theory to prove that any finitely presented group G has only finitely many subgroups of index 3.

8. Let X be the space obtained by attaching a torus $T = S^1 \times S^1$ to a cylinder $C = S^1 \times I$ via a homemorphism of the circle $S^1 \times \{(1, 0)\}$ of T with the circle $S^1 \times \{0\}$ of C. Compute the homology groups of X.