Math 871–872 Qualifying Exam

January 2012

Solve *three* problems from Section A and *three* more from Section B; you may work on any number of the problems, but indicate which six you want graded. When in doubt about the wording of a problem, ask for clarification. Do not interpret a problem in such a way that it becomes trivial. **Justify your answers**.

Section A:

(1) Recall that a continuous function $f : X \to Y$ between two topological spaces is called a *closed mapping* if for every closed subset C of X, $f(C)$ is a closed subset of Y.

Prove that if $f : X \to Y$ is continuous, X is compact, and Y is Hausdorff, then f is a closed mapping.

(2) Let X and Y be two topological spaces and let $X \times Y$ be endowed with the product topology.

(a) Prove that if X and Y each have a countable dense subset, then so does $X \times Y$.

(b) Prove that if X and Y are both connected, then so is $X \times Y$.

(3) Let $X = S^3$ (i.e., the subspace of $\mathbb{R}^3$ consisting of points (x, y, z) such that $x^2 + y^2 + z^2 = 1$). Define an equivalence relation on X by $(x, y, z) \sim (x', y', z')$ if and only if $z = z'$. Let $Y = X/\sim$ equipped with the quotient topology. Prove Y is homeomorphic to the closed interval $[-1, 1]$.

(4) On regularity.

(a) State the definition of what it means for a topological space to be regular. (Since there is some ambiguity in the literature about the definition of the term "regular", let us clarify that we want the definition of "regular" to include the Hausdorff property. Some use the term "regular Hausdorff" or "T_3" for the same thing.)

(b) Prove that a subspace of a regular space is also regular.

(c) Prove that a product of two regular spaces (equipped with the product topology) is also regular.

Section B:

(5) Show that every map from S^2 to the torus is null-homotopic. You may use without proof the fact that the fundamental group of S^2 is trivial.

(6) Let X be the quotient space of S^2 under the identification $x \sim -x$ for x in the equator.

(a) Write down a CW-decomposition of X.

(b) Write down the cellular chain complex for the CW structure from (a).
Be sure to include the differentials of the complex.

(c) Compute the homology of X.

(7) Let X be a space obtained by identifying the three vertices of a standard 2-simplex.

(a) Describe a structure of a Δ-complex on X.

(b) Write down the chain complex corresponding to the Δ-complex in (a).
Be sure to include the differentials of the complex.

(c) Compute the homology of X.

(8) Let $X = \mathbb{R}^3 - S^1$, the complement of a single circle in $\mathbb{R}^3$.

(a) Describe a deformation retraction of X onto $S^1 \vee S^2$.

(b) Compute $\pi_1(X)$.