Do *three* of the problems from section A and *three* from section B. If you work more than the required number of problems, make sure that you clearly mark which problems you want to have graded. If you have doubts about the wording of a problem or what results may be assumed without proof, ask for clarification. You may apply the statement of one of the problems below in your proof for a different problem, even if you do not include a proof the statement. Do not interpret a problem in such a way that it becomes trivial. **Justify your answers.**

Section A:

1. A topological space X has the $T_{2\frac{1}{2}}$ property if for each pair a, b of distinct points of X, there are open sets U, V of X such that $a \in U$, $b \in V$, and the closures satisfy $Cl_X(U) \cap Cl_X(V) = \emptyset$. Show that if A is a subspace of a $T_{2\frac{1}{2}}$ space, then A is also $T_{2\frac{1}{2}}$.

2. **(a)** Let $q : A \to B$ be continuous and $i : I \to I$ be the identity function. Define the function $q \times i : A \times I \to B \times I$ by $(q \times i)(a, t) := (q(a), i(t))$. Show that $q \times i$ is continuous.
Note: In part (b), you may assume the fact: If $q : A \to B$ is a quotient map, then so is $q \times i$.
(b) Let X be a path-connected Hausdorff space. The *cone* CX of X is the quotient of the space $X \times I$ by the smallest equivalence relation such that $(p, 0) \sim (q, 0)$ for all $p, q \in X$. Prove that there is a deformation retraction from CX to the singleton subspace $\{[(p, 0)]\}$.

3. Show that if $X = \mathbb{R}$ is the set of real numbers with the infinite ray topology $\mathcal{T}_X = \{(a, \infty) \mid a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$, then there are compact subsets C, D of X such that $C \cap D$ is not compact.

4. A space X is *locally connected* if for each $x \in X$ and each open set U containing x, there is a connected open set V in X satisfying $x \in V \subseteq U$. Let X and Y be locally connected spaces. Determine whether or not the product space $X \times Y$ must also be locally connected.

Section B:

5. Let $Y = (I \times I)/\sim$ be the quotient of the square using the equivalence relation generated by $(0, s) \sim (1, 1 - s)$ for all $s \in I$ and $(0, 0) \sim (0, 1)$. That is, Y is a Möbius band with two distinct points on its boundary glued together.
(a) Compute $\pi_1(Y)$.
(b) Describe a Δ-complex structure on Y, and use your complex to compute the simplicial homology groups of Y.

6. Let $p : \widetilde{X} \to X$ be a covering space with $\widetilde{X}$ path-connected. Let A be a subspace of X with inclusion map $i : A \hookrightarrow X$, and suppose that for $a \in A$ the induced homomorphism $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is a surjection. Show that if $r, s \in \widetilde{X}$ and $p(r) = p(s) = a$, then there is a path from r to s in the subspace $p^{-1}(A)$ of $\widetilde{X}$.

7. Use covering space theory to prove that if H is a finite index subgroup of a finitely presented group, then H is finitely presented.

8. Let X be a path-connected Hausdorff space. The *suspension* SX of X is the quotient of the space $X \times I$ by the smallest equivalence relation such that $(p, 0) \sim (q, 0)$ and $(p, 1) \sim (q, 1)$ for all $p, q \in X$. Prove that the homology groups satisfy $H_i(SX) = H_{i-1}(X)$ for all $i > 1$.