Math 871-872 Qualifying Exam
May 2017

Do three questions from Section A and three questions from Section B. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

Section A: Do THREE problems from this section.

(1) Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is injective and continuous, and $(Y, \mathcal{T}')$ is Hausdorff, then $(X, \mathcal{T})$ is Hausdorff.

(2) Suppose that $A, B \subseteq X$ are closed subsets of the topological space X, and both $A \cup B$ and $A \cap B$ are connected subsets of X. Show that both A and B are connected subsets of X.

(3) Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$, is continuous, f is surjective, X is compact, and Y is Hausdorff, then f is a quotient map.

(4) If X_i, $i \in \mathbb{N}$ are topological spaces and $x_i \in X_i$, let

$$Z = \{(y_i)_{i\in\mathbb{N}} \ : \ \text{for some } N \in \mathbb{N} \text{ we have } y_i = x_i \text{ for all } i \geq N\} \ .$$

Show that, in the product topology, Z is dense in $\prod_{i\in\mathbb{N}} X_i$.

Section B: Do THREE problems from this section.

(5) If $H : X \times [0, 1] \to X$ is a homotopy with $H_0 = H_1 =$ the identity map, show that the map $\gamma : I \to X$ given by $\gamma(t) = H(x_0, t)$ is a loop in X representing an element $g = [\gamma] \in \pi_1(X, x_0)$ which lies in the center of $\pi_1(X, x_0)$, i.e., $gh = hg$ for all $h \in \pi_1(X, x_0)$.

(6) Suppose that $X = U \cup V$ with U, V both open and $U \cap V, U$ and V path-connected, and $x_0 \in U \cap V$. Suppose also that the inclusion $i : U \cap V \to U$ induces a surjection $i_* : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$. Show that the inclusion $j : V \to X$ also induces a surjection $j_* : \pi_1(V, x_0) \to \pi_1(X, x_0)$.

(7) The free group $F(a, b)$ on two generators is the fundamental group of the one-point union $X = S^1 \vee S^1$ of two circles, using the common point $*$ as basepoint, with a and b corresponding to each of the loops S^1. Construct the covering space of X corresponding to the subgroup $H = \langle a^2b, aba, ab^2, bab, b^2a\rangle$ of $F(a, b)$, and determine the index of H in $F(a, b)$.

(8) Find a Δ-complex structure for, and compute the (simplicial) homology groups of, the space X obtained by gluing a 2-disk D^2 to the 2-sphere S^2 by a homeomorphism from ∂D^2 to the equator of S^2. That is,

$$X = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 \leq 1 \text{ and either } x^2 + y^2 + z^2 = 1 \text{ or } z = 0\} \ .$$

[Your description of a Δ-complex structure need only contain enough detail to describe your homology calculations.]