Math 871-872 Qualifying Exam
May 2019

**Do three questions from Section A and three questions from Section B**. You may work on as many questions as you wish, but you must indicate which ones you want graded. When in doubt about the wording of a problem or what results may be assumed without proof, ask for clarification. Do not interpret a problem in such a way that it becomes trivial.

**Section A: Do THREE problems from this section.**

A1. A subset $A$ of a topological space $(X, \mathcal{T})$ is *countably compact* if every countable open cover $\{\mathcal{U}_i\}_{i=1}^{\infty}$ of $A$ has a finite subcover.

(a) Show that the image of a countably compact set under a continuous map is countably compact.

(b) Show that a closed subset of a countably compact space is countably compact.

A2. Show that a topological space $(X, \mathcal{T})$ is *Hausdorff* if and only if the diagonal $\Delta = \{(x,x) : x \in X\}$ is a closed subset of $X \times X$, where $X \times X$ is equipped with the product topology.

A3. Recall that a topological space $X$ is *second countable* if there exists a countable basis for the topology of $X$, and $X$ is *separable* if it has a countable dense subset.

(a) Show that if $(X, \mathcal{T})$ has a metric topology, from the metric $d : X \times X \to \mathbb{R}$, and $X$ is separable, then $X$ is second countable.

(b) Show that $\mathbb{R}$, with the lower limit topology $\mathcal{T}_\ell$, is not metrizable, i.e., $\mathcal{T}_\ell$ is not the topology from any metric on $\mathbb{R}$.

A4. Suppose that $X$ and $Y$ are connected, nonempty topological spaces. Show that $X \times Y$ is also connected.

**Section B: Do THREE problems from this section.**

B1. Let $A$ be the annulus $\{re^{i\theta} \in \mathbb{C} : 1 \leq r \leq 2\}$, and let $B$ be the quotient space obtained from $A$ by identifying each point $e^{i\theta}$ on the circle $r = 1$ with the point $2e^{i(\theta+\pi)}$ on the circle $r = 2$. Determine $\pi_1(B)$.

B2. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be covering maps such that both $\widetilde{X}$ and $\widetilde{Y}$ are connected and simply-connected. Show that for every map $f : X \to Y$ there is a map $\tilde{f} : \widetilde{X} \to \widetilde{Y}$ such that $f \circ p = q \circ \tilde{f}$.

B3. Let $G$ be the free group generated by two elements, $G = \langle x, y \rangle$. Use covering space theory to prove that the commutator subgroup $[G, G]$ of $G$ (the normal subgroup generated by $xyx^{-1}y^{-1}$) is not finitely generated.

B4. For a topological space $X$, the *cone on* $X$, $cX$, is the quotient space of $X \times [0, 1]$ under the equivalence relation $(x, s) \sim (y, t)$ iff either $(x, s) = (y, t)$ or $s = t = 1$. The *suspension of* $X$, $\Sigma X$, is the union of two copies of $cX$ along $X \times \{0\}$, realized as the quotient space of $X \times [0, 1]$ under the equivalence relation $(x, s) \sim (y, t)$ iff either $(x, s) = (y, t)$ or $s = t = 1$ or $s = t = 0$.

(a) Show that the cone on $X$ is contractible.

(b) Use a Mayer-Vietoris sequence to show that for every $n \geq 1$ we have $\widetilde{H}_n(\Sigma X) \cong \widetilde{H}_{n-1}(X)$ .