Conformal Mapping: Further examples

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Overview

Conformal Map

We gather here several illustrations of conformal mappings. In certain cases we discuss the behavior of the map on the boundary of the relevant domain.

1. Translation, Dilation and Rotation

Translations and dilations provide the first simple examples. Indeed, if $h \in \mathbb{C}$, the translation $z \mapsto z + h$ is a conformal map from $\mathbb{C}$ to itself whose inverse is $w \mapsto w - h$. If h is real, then this translation is also a conformal map from the upper half-plane to itself. For any non-zero complex number c, the map $f : z \mapsto cz$ is a conformal map from the complex plane to itself, whose inverse is simply $g : w \mapsto c^{-1}w$. If c has modulus 1, so that $c = e^{i\phi}$ for some real ϕ then f is a rotation by ϕ. If $c > 0$ then f corresponds to a dilation. Finally, if $c < 0$ the map f consists of a dilation by $|c|$ followed by a rotation of π.

2. Power functions

$n \in \mathbb{N}$, $z \mapsto z^n$ is conformal from the sector $S = \{z \in \mathbb{C} : \ 0 < \arg(z) < \frac{\pi}{n}\}$ to $\mathbb{H}$ with the inverse $w \mapsto w^{\frac{1}{n}}$ defined in terms of the principal branch of the logarithm.

$0 < \alpha < 2$, the map $z \mapsto z^\alpha$ takes $\mathbb{H}$ to the sector $S = \{w \in \mathbb{C} : 0 < \arg(w) < \alpha\pi\}$. Indeed, if we choose the branch of the logarithm obtained by deleting the positive real axis and $z = re^{i\theta}$ with $r > 0$ and $0 < \theta < \pi$, then $f(z) = z^\alpha = r^\alpha e^{i\alpha\theta}$. Its inverse is given by $w \mapsto w^{\frac{1}{\alpha}}$. We need to choose the branch of the logarithm so that $0 < \arg(w) < \alpha\pi$.

Boundary behavior of f: If x travels from $-\infty$ to 0 on the real line, then $f(x)$ travels from $\infty e^{i\alpha\pi}$ to 0 on the half-line determined by $\arg(z) = \alpha\pi$. As x goes from 0 to ∞ on the real line, the image $f(x)$ goes from 0 to ∞ on the real line as well.

By composing the map just discussed with the translations and rotations in the previous example, we may map the upper half-plane $\mathbb{H}$ conformally to any (infinite) sector in $\mathbb{C}$.

3. Upper half disc to the first quadrant

The map $f(z) = \frac{1+z}{1-z}$ takes the upper half disc $\mathbb{D}^+ = \{z = x + iy : \ |z| < 1 \text{ and } y > 0\}$ conformally to the first quadrant $\mathbb{H}^+ = \{w = u + iv : \ u > 0 \text{ and } v > 0\}$. Indeed if $z = x + iy \in \mathbb{D}^+$,i.e., $x^2 + y^2 < 1$ and $y > 0$, then we have

$$f(z) = \frac{1 + x + iy}{1 - x - iy} = \frac{1 - (x^2 + y^2) + 2iy}{(1 - x)^2 + y^2} \in \mathbb{H}^+.$$

The inverse $g(w) = \frac{w-1}{w+1}$. is clearly holomorphic in the first quadrant. Moreover, $|w + 1| > |w - 1|$ for all w in the first quadrant because the distance from w to -1 is greater than the distance from w to 1; thus g maps into the unit disc.

$$g(w) = \frac{u + iv - 1}{u + iv + 1} = \frac{u^2 + v^2 - 1 + 2iv}{(u + 1)^2 + v^2} \in \mathbb{H}$$

if $v > 0$.

To examine the action of f on the boundary, note that if $z = e^{i\theta}$ $(0 < \theta < \pi)$ belongs to the upper half-circle, then

$$f(e^{i\theta}) = \frac{1+e^{i\theta}}{1-e^{i\theta}} = \frac{e^{-i\theta/2}+e^{i\theta/2}}{e^{-i\theta/2}-e^{i\theta/2}} = \frac{2\cos\frac{\theta}{2}}{-2i\sin\frac{\theta}{2}} = i\cot\frac{\theta}{2}.$$

AS θ travels from 0 to π, $f(e^{i\theta})$ travels along the imaginary axis from ∞ to 0. $f(x) = \frac{1+x}{1-x}$ is a bijection from $(-1,1)$ to $(0,\infty)$.

4. Logarithm $z \mapsto \log z$: $\mathbb{H}$ to Strip

The map $z \mapsto \log z$, defined as the branch of the logarithm obtained by deleting the negative imaginary axis, takes the upper half plane $\mathbb{H}$ to the strip $\{w = u + iv : \ u \in \mathbb{R}, \ 0 < v < \pi\}$. Let $z = re^{i\theta}$ with $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, then by definition,

$$\log z = \log r + i\theta.$$

The inverse map is then $w \mapsto e^w$. As x travels from $-\infty$ to 0, the point $f(x)$ travels from $\infty + i\pi$ to $-\infty + i\pi$ on the line $\{x + i\pi : \ -\infty < x < \infty\}$. When x travels from 0 to ∞ on the real line, its image $f(x)$ then goes from $-\infty$ to ∞ along the reals.

5. Logarithm $z \mapsto \log z$: Half disc to the half strip

$z \mapsto \log z$ also defines a conformal map from the half-disc $\{z = x + iy : \ |z| < 1, \ y > 0\}$ to the half-strip $\{w = u + iv : \ u < 0, \ 0 < v < \pi\}$. As x travels from 0 to 1 on the real line, then $\log x$ goes from $-\infty$ to 0. When x goes from 1 to -1 on the half-circle in the upper half-plane, then the point $\log z$ travels from 0 to πi on the vertical segment of the strip. Finally, as x goes from -1 to 0, the point $\log x$ goes from πi to $-\infty + \pi i$ on the top half-line of the strip.

6. Exponential $z \mapsto e^{iz}$: Half strip to half disc

The map $f(z) = e^{iz}$ takes the half-strip
$\{z = x + iy : \ -\frac{pi}{2} < x < \frac{\pi}{2}, \ y > 0\}$ conformally to the half-disc
$\{w = u + iv : \ |w| < 1, \ u > 0\}$. If $z = x + iy$, then

$$e^{iz} = e^{-y} e^{ix}.$$

Boundary behavior; If z goes from $\frac{\pi}{2} + i\infty$ to $\frac{\pi}{2}$, then $f(z)$ goes from 0 to i, and as x goes from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$, then $f(x)$ travels from i to $-i$ on the half-circle. Finally, as z goes from $-\frac{\pi}{2}$ to $-\frac{\pi}{2} + i\infty$, we see that $f(z)$ travels from $-i$ back to 0. The mapping f is closely related to the inverse of the map in Example 5.

7. $z \mapsto -\frac{1}{2}(z + \frac{1}{z})$: Half disc to upper half plane

The function $f(z) = -\frac{1}{2}(z + \frac{1}{z})$ is a conformal map from the half disk$\{z = x + iy : \ |z| < 1, \ y > 0\}$ to the upper half plane $\mathbb{H}$.
Boundary behavior: If x travels from 0 to 1, then $f(x)$ goes from ∞ to 1 on the real axis. If $z = e^{i\theta}$, then $f(e^{i\theta}) = -\cos\theta$ and as z travels from -1 to 1 along the unit half circle in the upper half-plane, $f(z)$ goes from 1 to -1 on the real segment. Finally, when x goes from -1 to 0, f(x) goes from 1 to $-infty$ along the real axis.

8. $\sin z$: the half strip to the upper half plane

The map $f(z) = \sin z$ takes the half-strip $\{w = x + iy : -\frac{\pi}{2} < x < \frac{\pi}{2},\ y > 0\}$ conformally onto the upper half-plane. If $\zeta = e^{iz}$, then

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} = \frac{1}{2}\left(-i\zeta + \frac{1}{-i\zeta}\right),$$

and therefore f is obtained first by applying the map in Example 6, then multiplying by $-i$ (that is, rotating by $-\frac{\pi}{2}$), and finally applying the map in Example 7. Boundary behavior: z travels from $-\frac{\pi}{2} + i\infty$ to $-\frac{\pi}{2}$, $f(z)$ goes from $-\infty$ to -1. When x goes from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, $f(x)$ goes from -1 to 1. Finally, if z goes from $\frac{\pi}{2}$ to $\frac{\pi}{2} + i\infty$, then $f(z)$ travels from 1 to ∞ on the real axis.