8.3.2 Montel's Theorem

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8150 Complex Analysis

March 30 - April 28, 2020

Overview

1. Necessary condition
2. Normal Family
3. The Proof of Montel's Theorem
4. Hurwitz's theorem

Necessary condition

What is the necessary condition on an open set Ω that guarantee the existence of a conformal map $F : \Omega \to \mathbb{D}$?

(1) If $\Omega = \mathbb{C}$, and $F : \Omega \to \mathbb{D}$, then F is entire and bounded. Hence F is constant. Hence $\Omega \neq \mathbb{C}$.

(2) $\mathbb{D}$ is connected $\implies \Omega$ is connected.

(3) $\mathbb{D}$ is simply connected $\implies \Omega$ is simply connected.

Theorem 1

Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists an unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Normal family

Definition 2

Let Ω be an open subset of $\mathbb{C}$. A family $\mathfrak{F}(\Omega)$ of holomorphic functions on Ω is said to be normal if every sequence in $\mathfrak{F}(\Omega)$ has a subsequence that converges uniformly on every compact subset of Ω (the limit need not be in $\mathfrak{F}(\Omega)$). The proof that a family of functions is normal is, in practice, the consequence of two related properties, *Uniform Boundedness and Equicontinuity*.

Definition 3

The family $\mathfrak{F}$ is said to be **uniformly bounded on compact subsets** of Ω if for each compact set $K \subset \Omega$ there exists $B > 0$, such that

$$|f(z)| \leq B \quad \text{for all} \quad z \in K \text{ and } f \in \mathfrak{F}.$$

The family $\mathfrak{F}$ is **equicontinuous** on a compact set $K \subset \Omega$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever $z, w \in K$ and $|z - w| < \delta$, then

$$|f(z) - f(w)| < \epsilon \quad \text{for all } f \in \mathfrak{F}.$$

Examples: 1)
$\{f_n(x) : [0,1] \to \mathbb{C}, \ |f_n'(x)| \leq M \text{ for some fixed constant } M\}$ in **uniform bounded and equicontinuity**.
2) $\{f_n(x) = x^n : x \in [0,1]\}$ is **uniform bounded** but not **equicontinuous**. note that $\lim_{n\to\infty} |f_n(1) - f_n(x_0)| = 1$ for $0 < x_0 < 1$.

Montel's Theorem

Theorem 4

Suppose $\mathfrak{F}(\Omega)$ is a family of holomorphic functions on Ω that is uniformly bounded on compact subsets of Ω. Then:

(i) *$\mathfrak{F}(\Omega)$ is equicontinuous on every compact subset of Ω.*

(ii) *$\mathfrak{F}(\Omega)$ is a normal family.*

The theorem really consists of two separate parts.

(1) $\mathfrak{F}$ is equicontinuous under the assumption that $\mathfrak{F}$ is a family of *holomorphic functions* that is uniformly bounded on compact subsets of Ω. The proof follows from an application of the Cauchy integral formula and hence relies on the fact that F consists of holomorphic functions. This conclusion is in sharp contrast with the real situation as illustrated by the family $\{f_n(x) = \sin nx : \; x \in (0,1)\}$, $|f_n(x)| \leq 1$, uniformly bounded; but not equicontinuous and has no convergent subsequences on any compact subinterval of $(0,1)$.

(2) The second part of the theorem is not complex-analytic in nature. Indeed, the fact that $\mathfrak{F}$ is a normal family follows from assuming only that F is uniformly bounded and equicontinuous on compact subsets of ω. This result is sometimes known as the **Arzela-Ascoli theorem** and its proof consists primarily of a *diagonalization argument*.

Exhaustion

A sequence $\{K_\ell\}_{\ell=1}^{\infty}$ of compact subsets of Ω is called an exhaustion if

(a) K_ℓ is contained in the interior of $K_{\ell+1}$ for all $\ell = 1, 2, \cdots$.

(b) Any compact set $K \subset \Omega$ is contained in K_ℓ for some ℓ. In particular $\Omega = \cup_{\ell=1}^{\infty} K_\ell$.

Lemma 5

Any open set Ω in the complex plane has an exhaustion.

Proof.

If Ω is bounded, we let $K_\ell = \{z \in \Omega : \ \mathrm{dist}(z, \partial\Omega) > \frac{1}{\ell}\}$.
If Ω is not bounded, we let $K_\ell = \{z \in \Omega : \ \mathrm{dist}(z, \partial\Omega) > \frac{1}{\ell} \text{ and } |z| < \ell\}$.
$\square$

The Proof of Montel's Theorem

Let K be a compact subset of Ω and choose $r > 0$ so small that $D_{3r}(z)$ is contained in Ω for all $z \in K$. It suffices to choose r so that $3r$ is less than the distance from K to the boundary of Ω. Let $z, w \in K$ with $|z - w| < r$, and let γ denote the boundary circle of the disc $D_{2r}(w)$. Then, by Cauchys integral formula, we have

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left[\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right] d\zeta$$

Observe that

$$\left| \frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right| = \frac{|z - w|}{|\zeta - z||\zeta - w|} \leq \frac{|z - w|}{r^2}.$$

since $\zeta \in \gamma$ and $|z - w| < r$.

Therefore

$$|f(z) - f(w)| \leq \frac{1}{2\pi} \frac{2\pi r}{r^2} B|z - w|,$$

where B denotes the uniform bound for the family $\mathfrak{F}$ in the compact set consisting of all points in Ω at a distance $\leq 2r$ from K. Therefore $|f(z) - f(w)| \leq C|z - w|$, and this estimate is true for all $z, w \in K$ with $|z - w| \leq r$ and $f \in \mathfrak{F}$, thus this family is equicontinuous.

To prove the second part of the theorem, we argue as follows. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence in $\mathfrak{F}$ and K a compact subset of Ω. Choose a sequence of points $\{w_j\}_{j=1}^{\infty}$ that is dense in Ω. Since $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded, there exists a subsequence $\{f_{n,1}\} = \{f_{1,1}, f_{2,1}, f_{3,1}, \cdots\}$ of $\{f_n\}$ such that $\{f_{n,1}(w_1)\}$ converges.

From $\{f_{n,1}\}$ we can extract a subsequence $\{f_{n,2}\} = \{f_{1,2}, f_{2,2}, f_{3,2}, \cdots\}$ so that $\{f_{n,2}(w_2)\}$ converges. We may continue this process, and extract a subsequence $\{f_{n,j}\}$ of $\{f_{n,j1}\}$ such that $\{f_{n,j}(w_j)\}$ converges.

Finally, let $g_n = f_{n,n}$ and consider the diagonal subsequence $\{g_n\}$. By construction, $\{g_n(wj)\}$ converges for each j, and we claim that equicontinuity implies that g_n converges uniformly on K. Given $\epsilon > 0$, choose δ as in the definition of equicontinuity, and note that for some J, the set K is contained in the union of the discs $D_\delta(w_1), \cdots, D_\delta(w_J)$. Pick N so large that if $n, m > N$, then

$$|g_m(w_j) - g_n(w_j)| < \epsilon \quad \text{for all } j = 1, 2, \cdots, J.$$

So if $z \in K$, then $z \in D_\delta(w_j)$ for some $1 \leq j \leq J$. Therefore,

$$\begin{aligned} & |g_n(z) - g_m(z)| \\ \leq & |g_n(z) - g_n(w_j)| + |g_n(w_j) - g_m(w_j)| + |g_m(w_j) - g_m(z)| \\ < & 3\epsilon \end{aligned}$$

whenever $n, m > N$. Hence $\{g_n\}$ converges uniformly on K.

Finally, we need one more *diagonalization* argument to obtain a subsequence that converges uniformly on every compact subset of Ω. Let $K_1 \subset K_2 \subset \cdots \subset K_\ell \subset \cdots$ be an exhaustion of ω, and suppose $\{g_{n,1}\}$ is a subsequence of the original sequence $\{f_n\}$ that converges uniformly on K_1. Extract from $\{g_{n,1}\}$ a subsequence $\{g_{n,2}\}$ that converges uniformly on K_2, and so on. Then, $\{g_{n,n}\}$ is a subsequence of $\{f_n\}$ that converges uniformly on every K_ℓ and since the K_ℓ exhaust Ω, the sequence $\{g_{n,n}\}$ converges uniformly on any compact subset of Ω, as was to be shown.

Hurwitz's theorem

Theorem 6

Suppose g_n are holomorphic and $g_n(z) \neq 0$ for all z in a region Ω. If g_n converges uniformly to g on compact subsets of Ω, then either g is identically zero in Ω or g is non-zero in Ω.

Proof.

The limit function g is holomorphic on Ω by *Weierstrasss Theorem*. In particular, if g is not identically zero, then the zeros of g are isolated. If $\gamma \sim 0$ is a simple curve on which $g \neq 0$, then by Weierstrasss Theorem, g_n' converges to g' and hence $\frac{g_n'}{g_n}$ converges to $\frac{g'}{g}$ uniformly on γ. By the Argument Principle, for n sufficiently large, the number of zeros of g enclosed by γ is the same as the number of zeros of g_n enclosed by γ, and since g_n is never zero, the theorem follows. □

Hurwitz's Corollary

Corollary 7

Suppose g_n are holomorphic and one-to-one for all z in a region Ω. If g_n converges uniformly to g on compact subsets of Ω, then either g is a constant in Ω or g is one-to-one in Ω.

Proof.

Fix a $w \in \Omega$ and apply the Hurwitz's theorem to $g - g(w)$ on $\Omega \setminus \{w\}$. □