8.3 Riemann Mapping Theorem

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Overview

Necessary condition

What is the necessary condition on an open set Ω that guarantee the existence of a conformal map $F : \Omega \to \mathbb{D}$?

(1) If $\Omega = \mathbb{C}$, and $F : \Omega \to \mathbb{D}$, then F is entire and bounded. Hence F is constant. Hence $\Omega \neq \mathbb{C}$.

(2) $\mathbb{D}$ is connected $\implies \Omega$ is connected.

(3) $\mathbb{D}$ is simply connected $\implies \Omega$ is simply connected.

Theorem 1

Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists an unique conformal map $F : \Omega \to \mathbb{D}$ such that $F(z_0) = 0$ and $F'(z_0) > 0$.

Proof of the corollary

Corollary 2

Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

Proof.

Suppose Ω_1 and Ω_2 are proper and simply connected. $F_1 : \Omega_1 \to \mathbb{D}$ and $F_2 : \Omega_2 \to \mathbb{D}$ are conformal mappings. Then $F = F_2^{-1} \circ F_1 : \Omega_1 \to \Omega_2$ are conformal and its inverse is given by $F^{-1} = F_!^{-1} \circ F_2 : \Omega_2 \to \Omega_1$. □

The Proof of the Uniqueness

Proof.

Suppose $F : \Omega \to \mathbb{D}$ and $G : \Omega \to \mathbb{D}$ are conformal and satisfy

$$F(z_0) = G(z_0) = 0 \quad \text{and} \quad F'(z_0) > 0, G'(z_0) > 0.$$

Then $H = F \circ G^{-1} : \mathbb{D} \to \mathbb{D}$ and $H(0) = F(G^{-1}(0)) = F(z_0) = 0$ and $H'(0) > 0$. This implies $H(z) = e^{i\theta} z$ and $H'(z) = e^{i\theta} > 0$. Hence $e^{i\theta} > 0$ and $e^{i\theta} = 1$. $H = F \circ G^{-1} = I$ and $F = G$ $\square$

The main idea of the proof of the existence: We consider all injective holomorphic maps $f : \Omega \to \mathbb{D}$ with $f(z_0) = 0$. From these we wish to choose an f so that its image fills out all of $\mathbb{D}$, and this can be achieved by making $f'(z_0) > 0$ as large as possible. In doing this, we shall need to be able to extract f as a limit from a given sequence of functions.

The main idea of the proof of the existence

We consider the family of $\mathfrak{F}$ of holomorphic functions on Ω defined as follows: $g \in \mathfrak{F}(\Omega)$ if and only if

(a) $g : \Omega \to D$ is holomorphic and one-to-one in Ω.

(b) $|g(z)| < 1$ all $z \in \Omega$.

(c) $g(z_0) = 0$ and $g'(z_0) > 0$.

We need to prove the following statements:

(A) $\mathfrak{F}(\Omega)$ is not empty.

(B) There exists a function $f \in \mathfrak{F}(\Omega)$ such that $g'(z_0) \leq f'(z_0)$ for all $g \in \mathfrak{F}$.

(C) If $f \in \mathfrak{F}(\Omega)$ satisfies (B), then f is the Riemann mapping, that is, $f(\Omega) = \mathbb{D}$, $f(z_0) = 0$ and $f'(z_0) > 0$.

The proof of (A)

Since $\Omega \neq \mathbb{C}$, we may take a point $a \notin \Omega$. Let $g(z) = \sqrt{z-a}$ be a branch of the square root function of $z - a \neq 0$, for $z \in \Omega$. $g(z)$ is holomorphic and one-to-one on Ω. Moreover , if g takes the value w in Ω, then it can not take the value $-w$. $g(\Omega)$ is open and there exists $r > 0$ with

$$D_r(g(z_0)) = \{w : \ |w - g(z_0)| < r\} \subset g(\Omega).$$

so we have

$$D_r(-g(z_0)) = \{w : \ |w + g(z_0)| < r\} \cap g(\Omega) = \emptyset.$$

Hence we have

$$|g(z) + g(z_0)| \geq r \quad \text{for all } z \in \Omega.$$

Then the function

$$g_1(z) = \frac{\epsilon}{g(z) + g(z_0)}$$

is holomorphic and one-to-one in Ω and satisfies $|g_1(z)| < 1$ for all $z \in \Omega$ provided that $|\epsilon| < r$.

Construction of an element in $\mathfrak{F}(\Omega)$

We can compose $g_1(z)$ with an automorphism ψ of the unit disc so that $g_0 = \psi \circ g_1(z)$ so that

$$g_0(z_0) = 0, \quad g_0'(z_0) > 0, \text{ that is } g_0 \in \mathfrak{F}(\Omega).$$

Here is how to choose the ψ:

$$\psi(z) = \psi_{g_1(z_0)}(z) = e^{i\theta} \frac{g_1(z_0) - z}{1 - \overline{g_1(z_0)} z}$$

$$\Longrightarrow g_0(z) = \psi \circ g_1(z) = e^{i\theta} \frac{g_1(z_0) - g_1(z)}{1 - \overline{g_1(z_0)} g_1(z)}$$

We have $g_0(z_0) = 0$.

Note that if $\psi_\alpha(z) = \frac{\alpha - z}{1-\bar{\alpha}z}$, then $\psi_\alpha'(z) = -\frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$. Then chain rule yields

$$g_0'(z) = -e^{i\theta}\frac{1-|g_1(z_0)|^2}{(1-\overline{g_1(z_0)}g_1(z))^2}g_1'(z).$$

This implies

$$g_0'(z_0) = -e^{i\theta}\frac{1-|g_1(z_0)|^2}{(1-|g_1(z_0)|^2)^2}g_1'(z_0) = -\frac{e^{i\theta}g_1'(z_0)}{1-|g_1(z_0)|^2}.$$

So we can choose θ so that $-e^{-i\theta} = \arg(g_1'(z_0))$. Therefore we have

$$g_0'(z_0) = \frac{|g_1'(z_0)|}{1-|g_1(z_0)|^2} > 0.$$

In the above construction, the only term that is not explicit is r. We determine r by the set Ω.

Proof of part (B)

By *Montel theorem*, $\mathfrak{F}(\Omega)$ is normal since $|f(z)| < 1$ for all $f \in \mathfrak{F}(\Omega)$. Let $M = \sup\{g'(z_0) : \ g \in \mathfrak{F}(\Omega)\}$. Note that $M \leq \infty$. Let $\{g_n(z)\} \subset \mathfrak{F}(\Omega)$ be a sequence of holomorphic function with $\lim_{n\to\infty} g_n'(z_0) = M$. Since $\mathfrak{F}(\Omega)$ is normal, there exists a subsequence $\{g_{n_k}(z)\}$ which uniformlly convergent on compact subsets of Ω, to a function $f(z)$; and in addition

$$\lim_{k\to\infty} g_{n_k}(z) = f(z) \quad \text{uniformly on compact subset of } \Omega.$$

In particular, $M < \infty$ and $f'(z_0) = M$. Then *Hurwitz theorem* implies $f(z)$ must be either one-to-one or constant because $g_{n_k}(z)$ is one-to-one. But $f'(z_0) = M > 0$, hence f is not constant. Moreover

$$f(z_0) = \lim_{k\to\infty} g_{n_k}(z_0) = 0 \text{ and } f'(z_0) = M > 0, \ f \ \text{ is one-to-one} \Longrightarrow f \in \mathfrak{F}(\Omega).$$

Since $f'(z_)) = M = \sup\{g'(z_0) : \ g \in \mathfrak{F}(\Omega)\}$, We have $g'(z_0) \leq f'(z_0)$ for all $g \in \mathfrak{F}(\Omega)$.

Proof of Part (C)

We will show that if the function f in part (B), then $f(\Omega) = \mathbb{D}$. Suppose not, then there would exist $w_0 \in \mathbb{D}$ such that $w_0 \notin f(\Omega)$. We will construct a function $g_1 \in \mathfrak{F}(\Omega)$ such that $g_1'(z_0) > f'(z_0)$. This will contradict to (B). The function

$$\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)} = -\psi_{w_0}(f(z))$$

is holomorphic and one-to-one from Ω into $\mathbb{D}$, and does not vanish on Ω since $w_0 \notin f(\Omega)$. Therefore there exists a branch of the square root

$$g(z) = \sqrt{\frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}} : \ \Omega \to \mathbb{D}, \text{ is one-to-one.}$$

Finally we normalized the function so that its derivative at z_0 is positive

$$g_1(z) = \frac{|g'(z_0)|}{g'(z_0)} \cdot \frac{g(z) - g(z_0)}{1 - \overline{g(z_0)}g(z)}$$

which has the same properties as $g(z)$, and in addition, in normalized so that $g_1(z_0) = 0$ and $g_1'(z_0) > 0$.

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The rest is some computation: Note that

$$\frac{d}{dz}\left(\frac{z-\alpha}{1-\bar{\alpha}z}\right) = \frac{1-|\alpha|^2}{(1-\bar{\alpha}z)^2}$$

and Chain rule yields

$$g_1'(z) = \frac{|g'(z_0)|}{g'(z_0)} \cdot \frac{1-|g(z_0)|^2}{1-\overline{g(z_0)}g(z)} \cdot g'(z)$$

Hence

$$g_1'(z_0) = \frac{|g'(z_0)|}{1-|g(z_0)|^2} > 0$$

Comparing $f'(z_0)$ and $g_1'(z_0)$

We first note that $f(z_0) = 0$, $f'(z_0) > 0$ and $(g(z_0))^2 = -w_0$.
Differentiate with respect to z of $(g(z))^2 = \frac{f(z) - w_0}{1 - \bar{w}_0 f(z)}$, we obtain

$$2g(z)g'(z) = \frac{1 - |w_0|^2}{(1 - \bar{w}_0 f(z))^2} \cdot f'(z)$$

Evaluate at $z = z_0$:

$$2g(z_0)g'(z_0) = (1 - |w_0|^2)f'(z_0) \implies |g'(z_0)| = \frac{1 - |w_0|^2}{2|g(z_0)|} f'(z_0).$$

Combine this and the last formula in previous page, we have

$$\begin{aligned} g_1'(z_0) &= \frac{|g'(z_0)|}{1 - |g(z_0)|^2} = \frac{1 - |w_0|^2}{2|g(z_0)|} \cdot \frac{f'(z_0)}{1 - |g(z_0)|^2} \\ &= \frac{1 - |w_0|^2}{2\sqrt{|w_0|}} \cdot \frac{f'(z_0)}{1 - |w_0|} \quad \text{here we have used } (g(z_0))^2 = -w_0 \\ &= \frac{1 + |w_0|}{2\sqrt{|w_0|}} f'(z_0) > f'(z_0) \quad \because \quad \frac{1 + |w_0|}{2\sqrt{|w_0|}} > 1 (\text{Schwartz inequality}) \end{aligned}$$