Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 1

Due 10 pm on 1/21/2021. Late homework will not be accepted

1. Describe geometrically the sets of points z in the complex plane defined by the following relations:

   (a) $|z-1|=1$. (b) $|z-1|=2|z-2|$. (c) $1/z=\bar{z}$.

   (d) $\mathrm{Re}(z)=3$ (e) $\mathrm{Im}(z)=a$ with $a\in\mathbb{R}$. (f) $\mathrm{Re}(z)>a$ with $a\in\mathbb{R}$.

   (g) $|z-1|<2|z-2|$.

2. Prove that $|z_1+z_2|\geq ||z_1|-|z_2||$ and explain when equality holds.

3. Prove that the equation $z^3+2z+4=0$ has its roots outside the unit circle. [Hint: what is the maximum value of the modulus of the first two terms if $|z|\leq 1$?]

4. (a) Prove that if $|w_1|=c|w_2|$ where $c>0$, then $|w_1-c^2w_2|=c|w_1-w_2|$.

   (b) Prove that if $c>0$, $c\neq 1$ and $z_1\neq z_2$, then $\left|\dfrac{z-z_1}{z-z_2}\right|=c$ represents a circle. Find its center and radius. [Hint: an easy way is to use part (a)]

5. (a) Let z,w be complex numbers, such that $\bar{z}w\neq 1$. Prove that

   $$\left|\frac{w-z}{1-\bar{w}z}\right|<1 \quad \text{if } |z|<1 \text{ and } |w|<1,$$

   and also that

   $$\left|\frac{w-z}{1-\bar{w}z}\right|=1 \quad \text{if } |z|=1 \text{ or } |w|=1.$$

   (b) Prove that for fixed w in the unit disk $\mathbb{D}:=\{z:\ |z|<1\}$, the mapping

   $$F: z\mapsto \frac{w-z}{1-\bar{w}z}$$

   satisfies the following conditions:

   (i) F maps $\mathbb{D}$ to itself and is holomorphic.

   (ii) F interchanges 0 and w, namely, $F(0)=w$ and $F(w)=0$.

   (iii) $|F(z)|=1$ if $|z|=1$.

   (iv) $F:\mathbb{D}\mapsto\mathbb{D}$ is bijective. [Hint: Calculate $F\circ F$.]

6. Use n-th roots of unity (i.e. solutions of $z^n-1=0$) to show that

   $$2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n}=n\ .$$

   [Hint: $1-\cos 2\theta=2\sin^2\theta,\ \sin 2\theta=2\sin\theta\cos\theta$.]

7. Prove that $f(z) = |z|^2$ has a derivative only at $z = 0$, but nowhere else.

8. Let $f(z)$ be analytic in a domain. Prove that $f(z)$ is a constant if it satisfies any of the following conditions:

   (a) $|f(z)|$ is constant;

   (b) $\mathrm{Re}(f(z))$ is constant;

   (c) $\arg(f(z))$ is constant;

   (d) $\overline{f(z)}$ is analytic;

   How do you generalize (a) and (b)?

9. Let $f(z)$ be analytic. Show that $\overline{f(\bar{z})}$ is also analytic.

10. (a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

   (b) Use these equations to show that the logarithm function defined by

$$\log z := \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

   is a holomorphic function in the region $r > 0$, $-\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.