Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 2

Due 10 pm on 1/28/2021. Late homework will not be accepted

1. Suppose $U(z)$ has continuous second order partial derivatives and $z = f(\zeta)$, $\zeta = \xi + i\eta$ is a holomorphic function. Show that

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} = |f'(\zeta)|^2 \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right] .$$

2. Show that $U(x^2 - y^2, 2xy)$ is harmonic if and only if $U(x, y)$ is.

3. Let $a_n \neq 0$ and assume that $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

4. Let f be a power series centered at the origin. Prove that f has a power series expansion around any point in its disc of convergence.

5. Prove the following:

   (a) The power series $\sum_{n=1}^{\infty} nz^n$ does not converge at any point of the unit circle.

   (b) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point of the unit circle.

   (c) The power series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges at every point of the unit circle except at $z = 1$.

6. Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^z}$$

defines a continuous function for $\mathrm{Re}(z) > 1$. (This is the Riemann ζ function and you will see later that it is analytic in the above region-not just being continuous there-and it can be extended to the complex plane with only 1 removed. You do not need to prove the latter assertions.) [Hint: Show that the series converges uniformly for $\mathrm{Re}(z) \geq 1 + \delta$, where $\delta > 0$ is any positive number.]

7. Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

has radius of convergence 1. Examine convergence at $z = 1,\ -1$ and i.

8. It was defined that $e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Show that $e^z = \lim_{n\to\infty} \left(1 + \frac{z}{n}\right)^n$.

9. (a) Express $\sin z$ in the form $u + iv$. Do the same for $\cos z$.

   (b) Show that there exists a sequence z_n such that $\sin z_n \to \infty$. Do the same for $\cos z$. (This shows that unlike their real counter parts, $\sin z$ and $\cos z$ are unbounded.)

10. Prove that

$$\text{(a)} \lim_{z\to(2k+1)\pi/2} \tan z = \infty\ , \quad \text{(b)} \lim_{z\to(2k+1)\pi/2} [z-(2k+1)\pi/2] \tan z = -1\ .$$

   [Hint: The proof of (b) is *very short* if done the "right" way.]

11. Show that if $|\alpha| < r < |\beta|$, then

$$\int_\gamma \frac{1}{(z-\alpha)(z-\beta)} = \frac{2\pi i}{\alpha - \beta}$$

   where γ denotes the circle centered at the origin, of radius r, with positive orientation.

12. Assume f is continuous in the region: $x \geq x_0,\ 0 \leq y \leq b$ and the limit

$$\lim_{x\to+\infty} f(x+iy) = A$$

   exists uniformly with respect to y (independent of y). Show that

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

   where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.