Mathematics Department
The University of Georgia
Math 8150 Homework Assignment 3

Due 10 pm on 2/9/2021. Late homework will not be accepted

1. Assume real value functions u, v of two variables have continuous partial derivatives at (x_0, y_0). Show that $f = u + iv$ has derivative $f'(z_0)$ at $z_0 = x_0 + iy_0$ if and only if

$$\lim_{r\to 0} \frac{1}{\pi r^2} \int_{|z-z_0|=r} f(z)\, dz = 0 \ .$$

2. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

3. Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

4. Suppose that $f(z)$ is entire and $\lim_{z\to\infty} f(z)/z = 0$. Show that $f(z)$ is a constant.

5. Let f be analytic on a domain D and let γ be a closed curve in D. For any z_0 in D not on γ, show that

$$\int_\gamma \frac{f'(z)}{(z - z_0)} dz = \int_\gamma \frac{f(z)}{(z - z_0)^2} dz$$

Give a generalization of this result.

6. Compute $\displaystyle\int_{|z|=1} \left(z + \frac{1}{z}\right)^{2n} \frac{dz}{z}$ and use it to show that

$$\int_0^{2\pi} \cos^{2n}\theta d\theta = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

7. Prove that

$$\int_0^\infty \sin(x^2)\, dx = \int_0^\infty \cos(x^2)\, dx = \frac{\sqrt{2\pi}}{4}.$$

These are the Fresnel integrals. Here $\displaystyle\int_0^\infty$ is interpreted as $\displaystyle\lim_{R\to\infty} \int_0^R$.

[For a Hint, see textbook by Stein and Shakarchi p64, #1]

8. Show that $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

   [For a Hint, see textbook by Stein and Shakarchi p64, #2]

9. Goursat's theorem under the additional hypothesis that f' is continuous:

   Suppose $f'(z)$ is continuous on a region Ω, and that the triangle T along with its interior is in Ω. Apply Green's theorem to show that

   $$\int_T f(z)\,dz = 0.$$

   [For a Hint, see textbook by Stein and Shakarchi p65, #5]

10. Suppose $f'(z)$ exists in a region Ω except possibly at point w in Ω. Let $T \subset \Omega$ be a triangle whose interior is also contained in Ω and contains w. Show that

    $$\int_T f(z)\,dz = 0.$$

11. Let f be a holomorphic function on the strip $-1 < y < 1$, $x \in \mathbb{R}$ such that

    $$|f(z)| \leq A(1+|z|)^\eta, \quad \eta \text{ is a fixed real number}$$

    for all z in the strip. Show that for each integer $n \geq 0$, there exists A_n such that

    $$|f^{(n)}(x)| \leq A_n(1+|x|)^\eta, \quad \text{for all } x \in \mathbb{R}.$$

    [Hint: Use Cauchy inequalities]