Math 8155 Starter Problems

Complex Numbers and Plane

1. Use de Moivre's theorem (i.e. $(e^{i\theta})^n == \cos n\theta + i\sin n\theta$, or $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$) to find the sum
$$\sin x + \sin 2x + \cdots + \sin nx$$

2. Let z_k $(k = 1, \cdots, n)$ be complex numbers lying on the same side of a straight line passing through the **origin**. Show that
$$z_1 + z_2 + \cdots + z_n \neq 0, \quad 1/z_1 + 1/z_2 + \cdots + 1/z_n \neq 0.$$
Hint: Consider a special situation first.

3. Let $f(z) = z + 1/z$. Describe the images of both the circle $|z| = r$ of radius r $(r \neq 0)$ and the ray $\arg z = \theta_0$ under f in terms of well known curves.

4. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$ for any two complex numbers z_1, z_2, and explain the geometric meaning of this identity.

5. (1). Characterize positive integers n such that $(1 + i)^n = (1 - i)^n$.

   (2). Let n be a natural number. Show that
$$[1/2(-1 + \sqrt{3}i)]^n + [1/2(-1 - \sqrt{3}i)]^n$$
is equal to 2 if n is a multiple of 3, and it is equal to -1 otherwise.

6. Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that
$$\cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \cdots + \cos\frac{2(n-1)\pi}{n} + = -1 \text{ and}$$
$$\sin\frac{2\pi}{n} + \sin\frac{4\pi}{n} + \sin\frac{6\pi}{n} + \cdots \frac{2(n-1)\pi}{n} = 0$$
Hint: If $z^n + c_1 z^{n-1} + \cdots + c_{n-1}z + c_n = 0$ has roots z_1, z_2, ..., z_n, then
$$z_1 + z_2 + \cdots + z_n = -c_1,$$
$$z_1 z_2 \cdots z_n = (-1)^n c_n \text{ (not used).}$$

7. Describe each set in the z-plane in (a) and (b) below, where α is a complex number and k is a positive number such that $2|\alpha| < k$.

   (a) $|z - \alpha| + |z + \alpha| = k$;

   (b) $|z - \alpha| + |z + \alpha| \leq k$.