Qualifying Exam 970/953 January 20, 2004

Instructions: Do three problems from Part I, and three problems from Part II. If you do more than three problems from either section, you must indicate which three you want to count.

Part I: Topology (970)

1. (a) Suppose that $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ are topological spaces, $A \subseteq X$, and $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homeomorphism. Show that $f|_A : (A, \mathcal{T}_A) \to (f(A), \mathcal{T}'_{f(A)})$ is also a homeomorphism (using the corresponding subspace topologies).

   (b) Show that there is no homeomorphism $f : [0, 1) \to (0, 1)$ (where each has the subspace topology it inherits from the usual topology on $\mathbf{R}$).

2. Show that if the metric space (X, d) is separable (i.e., it contains a countable dense subset), then the metric topology on X is second countable.

3. Let $(X, \mathcal{T})$ be a Hausdorff space and let $\mathcal{T}' = \{\mathcal{U} \subseteq X : X \setminus \mathcal{U} \subseteq X \text{ is compact}\} \cup \{\emptyset\}$. Show that $\mathcal{T}'$ is a topology on X, and is coarser than $\mathcal{T}$. Show that, in general, they need not be equal.

4. Two subsets $A, B \subseteq X$ of the space $(X, \mathcal{T})$ are called *separated* if there are $U, V \in \mathcal{T}$ with $A \subseteq U \subseteq X \setminus B$ and $B \subseteq V \subseteq X \setminus A$. We say X is *completely normal* if X is T_1 and if for every pair of separated subsets A, B, there are $U, V \in \mathcal{T}$ so that $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Show that a space $(X, \mathcal{T})$ is completely normal $\iff$ every subset of X is normal.

5. Show that if $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ is a homotopy equivalence, then f induces a bijective correspondence between the path-components of X and the path-components of Y .

Part II: Algebraic Geometry (953)

1. Prove that the ideal $(x^n + y^n + z^n) \subset \mathbf{C}[x, y, z]$ is a prime ideal for every integer $n > 0$.

2. Find the multiplicity of each singular point of the zero locus of $xyz(x + y) - x^4 - y^4$ in $\mathbf{CP}^2$.

3. Let $C \subset \mathbf{CP}^2$ be the zero locus of a homogeneous polynomial of degree 5. Assume C is smooth. Let D be a divisor on C of degree 11. What is the most you can say about $\dim \Gamma(D)$? (For example, is it positive? Can you give a lower bound? Can you give an upper bound? Can you determine it exactly?) Justify your answer.

4. Let $\mathcal{F}$ be the structure sheaf of $\mathbf{CP}^2$, and let $\mathcal{I}$ be the sheaf of ideals defining two distinct reduced points p, q of $\mathbf{CP}^2$. Let $\mathcal{G}$ be the presheaf defined for any Zariski open set $U \subset \mathbf{CP}^2$ as $\mathcal{G}(U) = \mathcal{F}(U)/\mathcal{I}(U)$.

   (a) Show that $\mathcal{G}$ is not a sheaf.

   (b) Determine the associated sheaf $\mathcal{G}^+$ (i.e., for any Zariski open set $U \subset \mathbf{CP}^2$, determine $\mathcal{G}^+(U)$).

5. (a) Show that the closed points of any scheme of finite type over a field are dense.

   (b) Give an example of a nonempty scheme whose closed points are not dense.