Fall 2019 Algebra Qualifying Exam Solutions

1. Let V be a 5-dimensional vector space over a field F.

   (a) Let $T : V \to V$ be a linear transformation with characteristic polynomial $(x-1)^3(x-2)^2$ and minimal polynomial $(x-1)^2(x-2)$.

   i. Write down a matrix which represents T in Jordan normal form. 3 points

   ii. Write down the matrix which represents T in rational normal form. 7 points

   (b) Instead, let $T : V \to V$ be a *nilpotent* linear transformation which has exactly one 2-dimensional invariant subspace.

   i. How many similarity classes of such linear maps T are there? 3 points

   ii. Assuming finally that F is the finite field $\mathbb{F}_q$ with q elements, find an explicit formula for the number of such linear maps T. 7 points

(a)

i. $$\begin{pmatrix} 1 & 1 & & 0 & \\ & 1 & 0 & & \\ & & 1 & & \\ 0 & & & 2 & 0 \\ & & & & 2 \end{pmatrix}$$

ii. This is the $F[x]$-module $F[x]/(x-1) \oplus F[x]/(x-1)^2 \oplus F[x]/(x-2) \oplus F[x]/(x-2)$

$$\overset{CRT}{\cong} F[x]/\underbrace{(x-1)(x-2)}_{= x^2-3x+2} \oplus F[x]/\underbrace{(x-1)^2(x-2)}_{= x^3-4x^2+5x-2}$$

invariant factors

So: $$\begin{pmatrix} 0 & -2 & & & \\ 1 & 3 & & 0 & \\ & & 0 & 0 & 2 \\ & 0 & 1 & 0 & -5 \\ & & 0 & 1 & 4 \end{pmatrix}$$

(b)

i. <u>One</u> – it is a single Jordan block
$$\begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & 1 & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix}$$

ii. Orbit size = index of centralizer in $GL_5(\mathbb{F}_q)$

invertible matrices
$$\begin{pmatrix} a & b & c & d & e \\ 0 & a & b & c & d \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & a \end{pmatrix}$$

order $q^{10}(q^5-1)(q^4-1)(q^3-1)(q^2-1)(q-1)$

order $(q-1)q^4$

$\therefore$ its $q^6(q^5-1)(q^4-1)(q^3-1)(q^2-1)$

2. Let A be a finite-dimensional algebra over a field F equipped with a non-degenerate symmetric bilinear form $(\cdot,\cdot) : A \times A \to F$. Let $x_1, \ldots, x_n$ be a basis for A and $y_1, \ldots, y_n$ be the dual basis with respect to the given form, i.e., $(x_i, y_j) = \delta_{i,j}$ for all $i, j = 1, \ldots, n$.

   (a) Show that the element
   $$z := \sum_{i=1}^{n} x_i y_i$$
   is well defined independent of the initial choice of the basis $x_1, \ldots, x_n$.

   (b) Assume for the remainder of the question that the form $(\cdot,\cdot)$ is *invariant*, which means that $(ab, c) = (a, bc)$ for all $a, b, c \in A$. Show that $([a, b], c) = (a, [b, c])$ where $[\cdot,\cdot]$ is the commutator.

   (c) Let $a \in A$ be any element and suppose that $[a, x_i] = \sum_{j=1}^{n} \lambda_{ij} x_j$ and $[a, y_i] = \sum_{j=1}^{n} \mu_{ij} y_j$ for scalars $\lambda_{ij}, \mu_{ij} \in F$. Show that $\lambda_{ij} + \mu_{ji} = 0$. Deduce that z lies in the *center* of the algebra A. (You may find the identity $[a, xy] = [a, x]y + x[a, y]$ helpful here.)

6 points

4 points

5+5 points

(a) Let $x_1', \ldots, x_n'$ be another basis with dual basis $y_1', \ldots, y_n'$.

Say $x_j' = \sum_i a_{ij} x_i$, $y_i = \sum_j b_{ij} y_j'$.

$\Rightarrow (x_j', y_i) = a_{ij} = b_{ij}$

Then $z = \sum_i x_i y_i = \sum_{i,j} b_{i,j} x_i y_j'$

$z' = \sum_j x_j' y_j' = \sum_{i,j} a_{ij} x_i y_j'$

the same!

(b) $([a,b],c) = (ab - ba, c) = (ab,c) - (c, ba)$

$= (a, bc) - (cb, a) = (a, bc - cb)$

$= (a, [b,c])$ //

(c)

$\lambda_{ij} = ([a, x_i], y_j) = -([x_i, a], y_j) = -(x_i, [a, y_j])$

$\mu_{ji} = (x_i, [a, y_j])$

sum to $\lambda_{ij} + \mu_{ji} = 0$

To show z central, show $[a, z] = 0 \quad \forall a$.

$$[a, z] = \sum_i [a, x_i y_i] = \sum_i \left([a, x_i] y_i + x_i [a, y_i] \right)$$

$$= \sum_{i,j} \lambda_{ij} x_j y_i + \sum_{i,j} x_i \mu_{ij} y_j$$

$$= \sum_{i,j} (\lambda_{ij} + \mu_{ji}) x_j y_i = 0$$

3. In this question, R is a ring and $e \in R$ is an idempotent, so that eRe is another ring with identity element e.

   (a) What does it mean to say that R is *semisimple*? State the Artin-Wedderburn Theorem. 4 points

   (b) If V is a completely reducible R-module of finite length, show that the algebra $\mathrm{End}_R(V)$ is semisimple. Deduce for a semisimple ring R that eRe is semisimple too. 4+4 points

   (c) Assuming that R is left Artinian, show that $J(eRe) = eJ(R)e$, where J denotes Jacobson radical. 8 points

(a) R is semisimple if ${}_R R$ is completely reducible module.

AW: R is semisimple $\iff$ $R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$

for $r \geq 0$, $n_1, \ldots, n_r \geq 1$, division algebras $D_1, \ldots, D_r$.

(could write $\oplus$ instead of $\times$, okay too).

(b) $V \cong L_1^{n_1} \oplus \cdots \oplus L_r^{n_r}$. $L_1, \ldots, L_r$ pairwise $\not\cong$ irreps.

Let $D_i = \mathrm{End}_R(L_i)$ — a division algebra by Schur's Lemma.

As $\mathrm{Hom}_R(L_i, L_j) = 0$ for $i \neq j$,

$\mathrm{End}_R(V) \cong \mathrm{End}_R(L_1^{n_1}) \oplus \cdots \oplus \mathrm{End}_R(L_r^{n_r})$

$\cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$ which is semisimple

For eRe, note $eRe \cong \mathrm{End}_R(Re)^{op}$.

As R is semisimple, $Re <_R R$ is c.r. of finite length, so

$\mathrm{End}_R(Re)^{op}$ is semisimple by previous part.

You can remove the op — R is semisimple $\iff$ R^{op} is semisimple

(c) In an Artinian ring, $J(R)$ is nilpotent 2-sided ideal such that $R/J(R)$ is semisimple (characterisation)

To use this for question, need to check that R Artinian $\Rightarrow$ eRe Artinian.

[Let $J_1 \supset J_2 \supset \cdots$ be a chain of (left) ideal in eRe.

Then $RJ_1 \supset RJ_2 \supset \cdots$ is one in R, so stabilises

$RJ_n = RJ_{n+1} = \cdots$

Now multiply by e:

$eRJ_n = eRJ_{n+1} = \cdots$

But $eRJ_n = eReJ_n = J_n$, so this does the job: $J_n = J_{n+1} = \cdots$]

As $J(R)$ is nilpotent 2-sided ideal of R, $eJ(R)e$ is so in eRe.

Remains to show $\dfrac{eRe}{eJ(R)e}$ is semisimple.

$$\cong \quad \bar{e}\underbrace{\left(R/J(R)\right)}_{\text{semisimple}}\bar{e} \quad \text{where } \bar{e} = e + J(R) \in R/J(R)$$

This is indeed semisimple by (b)

4. Let G be a finite group. Adopt the usual notation for the character table of G. In particular, $C_1 = \{1\}, C_2, \ldots, C_n$ are the conjugacy classes and $\chi_1 = \mathbf{1}, \chi_2, \ldots, \chi_n$ are the irreducible characters.

(a) Let $\rho : G \to GL_n(\mathbb{C})$ be a finite-dimensional representation with associated character χ. Prove that $\ker \rho = \{g \in G \mid \chi(g) = \chi(1)\}$.

6 points

(b) Use the row and column orthogonality relations to work out the values of α, β, γ and δ in the following character table:

6 points

	C_1	C_2	C_3	C_4	C_5	C_6	C_7	C_8	C_9
#	1	1	20	30	12	12	20	12	12
χ_1	1	1	1	1	1	1	1	1	1
χ_2	2	-2	-1	γ	$-\beta$	$-\alpha$	1	α	β
χ_3	2	-2	-1	γ	$-\alpha$	$-\beta$	1	β	α
χ_4	3	3	0	-1	β	α	0	α	β
χ_5	3	3	0	-1	α	β	0	β	α
χ_6	4	-4	1	γ	-1	-1	-1	1	1
χ_7	4	4	1	γ	-1	-1	1	-1	-1
χ_8	5	5	-1	1	0	0	-1	0	0
χ_9	δ	$-\delta$	0	γ	1	1	0	-1	-1

(c) Let G be a group with the character table computed in (b). Work out the character table of the group $H = G/Z(G)$, explaining your steps. What group is H?

6+2 points

(a) Take $g \in G$. As $g^N = 1$ some N, the min. poly. of $\rho(g)$ divides $x^N - 1$, which has distinct linear factors.

Hence, $\rho(g)$ is diagonalisable, say to $\begin{pmatrix} c_1 & & 0 \\ & \ddots & \\ 0 & & c_n \end{pmatrix}$,

each c_i is a root of unity.

So $\chi(g) = c_1 + \cdots + c_n \xleftarrow{\triangle \text{ ineq.}} |c_1 + \cdots + c_n| \leq |c_1| + \cdots + |c_n| = n$,
equality $\iff c_1 = \cdots = c_n$

Now $\chi(g) = \chi(1) \iff c_1 + \cdots + c_n = n$ $\Big\}$ $\triangle$ ineq.

$\iff c_1 = \cdots = c_n = 1$

$\iff \rho(g) = I$

$\iff g \in \ker \rho$

(b) Note α, β are real :—

by def. of inner product

$$(\chi_2 \chi_2^*, \chi_1) = (\chi_2, \chi_2) = 1$$

$\therefore$ $\chi_2\chi_2^* - \chi_1$ is a character of degree 3 with no trivial constituents, so could only be χ_4 or χ_5 (using $\delta \neq 1$, else what's $\chi_2\chi_9$?)

Also <u>real-valued</u>, so α, β are real as values of χ_4 / χ_5

Now dot columns together using this to see :

$0 = C_1 \cdot C_2 \Rightarrow \delta^2 = 36 \Rightarrow$ $\boxed{\delta = 6}$

$0 = C_4 \cdot C_7 \Rightarrow 2\gamma = 0 \Rightarrow$ $\boxed{\gamma = 0}$

$0 = C_4 \cdot C_5 \Rightarrow \alpha + \beta = 1$

$0 = C_8 \cdot C_9 \Rightarrow \alpha\beta = -1$

$\therefore \alpha, \beta$ are roots of $x^2 - x - 1$

so $\boxed{\frac{1 \pm \sqrt{5}}{2} \quad \alpha, \beta}$

Note now we can also find the conjugacy class sizes. See table..

(c) $Z(G) =$ classes of size 1

$\therefore Z(G) = C_1 \cup C_2$, size 2, while $|G| = 120$

So $|G/Z(G)| = 60$.

If $C_2 = \{z\}$, $z^2 = 1$, so it's either $+1$ or -1 on each irrep.

The irreps of $G/Z(G)$ are same as ones of G on which z is $+1$

Classes in $G/Z(G)$ are either images of 1 or 2 classes in G ...

	$C_1 \cup C_2$	$C_3 \cup C_7$	C_4	$C_5 \cup C_9$	$C_6 \cup C_8$
χ_1	1	1	1	1	1
χ_4	3	0	-1	β	α
χ_5	3	0	-1	α	β
χ_7	4	1	0	-1	-1
χ_8	5	-1	1	0	0

Observe its a simple group of order 60

So it must be A_5

5. The dihedral group $G = \langle a, b \mid a^3 = b^2 = 1, bab = a^{-1} \rangle$ acts on the $\mathbb{C}$-algebra $S = \mathbb{C}[x, y]$ by algebra automorphisms so that

$$a \cdot x = \omega x, \qquad a \cdot y = \omega^{-1} y, \qquad b \cdot x = y,$$

where $\omega = e^{2\pi i/3}$. Let $R := S^G$ be the invariant subalgebra.

(a) Show that $R = \mathbb{C}[x^3 + y^3, xy]$. 10 points

(b) Show that $R \subseteq S$ is an integral extension. 5 points

(c) Find an explicit monic polynomial $f(t) \in R[t]$ such that $f(x) = 0$. 5 points

(a) Consider $z = \sum c_{ij} x^i y^j \in S$

$$a \cdot z = \sum c_{ij} \omega^{i-j} x^i y^j$$

So need $c_{ij} = 0$ unless $i \equiv j \pmod 3$

$$b \cdot z = \sum c_{ji} x^i y^j$$

So need $c_{ij} = c_{ji}$ $\forall i, j$.

Show R is spanned by $x^3 + y^3$ and xy

(you can get any $x^{3n} + y^{3n}$ as monomials in these!)

$\therefore R = \mathbb{C}[x^3 + y^3, xy]$

(b) This is a general fact about invariants of finite groups.

Given $z \in S$, consider $f(t) = \prod_{g \in G} (t - g \cdot z)$

It's monic in $R[t]$ with z as a root $\Rightarrow$ integral.

(c) Use the recipe from (b)!

$$f(t) = \prod_{g \in G} (t - g \cdot x)$$

$$= (t-x)(t-\omega x)(t-\omega^2 x)(t-y)(t-\omega y)(t-\omega^2 y)$$

$$= (t^3 - x^3)(t^3 - y^3)$$

$$= t^6 - (x^3 + y^3)\, t^3 + x^3 y^3$$

6. Work over an algebraically closed field F of characteristic zero.

(a) Let X be an affine variety with coordinate algebra $F[X]$. State the *Nullstellensatz*. Then use it to show that a subset $S \subseteq X$ is dense (in the Zariski topology) if and only if the following property holds for all $f \in F[X]$: 5+5 points

$$(f(s) = 0 \text{ for all } s \in S) \Rightarrow f = 0.$$

(b) Given affine varieties X and Y and dense subsets $S \subseteq X$ and $T \subseteq Y$, prove that $S \times T$ is dense in $X \times Y$. 6 points

(c) Show that the integer lattice $\mathbb{Z}^n$ is dense in F^n. 4 points

(a) NSS:

$$\left\{ \begin{array}{c} \text{radical ideals} \\ \text{of } F[X] \end{array} \right\} \underset{I}{\overset{V}{\rightleftarrows}} \left\{ \begin{array}{c} \text{closed subsets} \\ \text{of } X \end{array} \right\}$$

$V(J) = \{ x \in X \mid f(x) = 0 \ \forall f \in J \}$

$I(S) = \{ f \in F[X] \mid f(x) = 0 \ \forall x \in S \}$

These maps are mutually inverse, inclusion reversing bijections.

For second part, we need to show $\overline{S} = X \iff I(S) = (0)$

<u>Claim</u> $\overline{S} = V(I(S))$.

<u>Pf</u>. $S \subseteq V(I(S))$ closed hence $\overline{S} \subseteq V(I(S))$.

If $S \subseteq V(J')$ for $J' = \sqrt{J'}$, $V(I(S)) \subseteq V(I(V(J'))) = V(J')$

This shows $V(I(S)) \subseteq \overline{S}$ //

Hence, $\bar{S} = X \iff V(I(S)) = X$

$$\iff I(S) = I(X) = (0)$$

(b) There are several proofs. All should note somewhere that $F[X \times Y] = F[X] \otimes F[Y]$ by def. of product.

Say $\theta \in F[X \times Y]$ is zero on $S \times T$.

RTP $\theta = 0$.

Write $\theta = \sum_i f_i \otimes g_i \in F[X] \otimes F[Y]$, f_i's lin. ind.

For any $t \in T$, $\sum_i f_i\, g_i(t) \in F[X]$ is zero on S,

hence zero as S is dense in X.

As f_i's are linearly independent, this implies $g_i(t) = 0 \ \forall t \in T$,

so $g_i = 0$ as T is dense in Y

$\therefore \theta = 0$

(c) This follows from (b) by induction on n as $\mathbb{Z}$ is dense in F (being infinite!).