ALGEBRA HW 4

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3.2.19

Prove that if N is a normal subgroup of the finite group G and $(|N|, |G : N|) = 1$ then N is the unique subgroup of G of order $|N|$.

Proof. Let $H \leq G$ such that $|H| = |N|$. By Proposition 13,

$$|NH| = \frac{|N||H|}{|N \cap H|} = \frac{|N|^2}{|N \cap H|} = |N|\frac{|N|}{|N \cap H|}$$

Since N is normal, $NH \leq G$ by Corollary 15, so $|NH|$ divides $|G|$, or

$$|G| = m|NH| = m|N|\frac{|N|}{|N \cap H|}.$$

On the other hand,

$$|G| = |G : N||N|,$$

so

$$|N||G : N| \quad = |N|m\tfrac{|N|}{|N \cap H|} \quad \text{cancelling yields}$$
$$|G : N| = m\tfrac{|N|}{|N \cap H|}.$$

Hence, $\frac{|N|}{|N \cap H|}$ divides both $|N|$ and $|G : N|$ so, by hypothesis,

$$\frac{|N|}{|N \cap H|} = 1$$

Therefore, $|N \cap H| = |N|$, which means, since $|H| = |N|$, that $H = N$. We conclude, then, that N is the unique subgroup of G with order $|N|$. □

3.4.2

Find all 3 composition series for Q_8 and all 7 composition series for D_8. List the composition factors in each case.

Answer:

$$1 \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, i, -i\} \trianglelefteq Q_8$$
$$1 \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, j, -j\} \trianglelefteq Q_8$$

and

$$1 \trianglelefteq \{1, -1\} \trianglelefteq \{1, -1, k, -k\} \trianglelefteq Q_8$$

where, in each case, $N_{i+1}/N_i = \mathbb{Z}/2\mathbb{Z}$.

$$1 \trianglelefteq \langle s \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle s, r^2 \rangle \trianglelefteq D_8$$

$$1 \trianglelefteq \langle r^2 \rangle \trianglelefteq \langle r \rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle sr \rangle \trianglelefteq \langle sr, sr^3 \rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle sr^3 \rangle \trianglelefteq \langle sr, sr^3 \rangle \trianglelefteq D_8$$
$$1 \trianglelefteq \langle sr^2 \rangle \trianglelefteq \langle s, sr^2 \rangle \trianglelefteq D_8$$

and

$$1 \trianglelefteq \langle s \rangle \trianglelefteq \langle s, sr^2 \rangle \trianglelefteq D_8$$

where, in each case, $N_{i+1}/N_i = \mathbb{Z}/2\mathbb{Z}$.

♣

3.4.5

Prove that subgroups and quotient groups of a solvable group are solvable.

Proof. Let G be a solvable group with composition series

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \ldots \trianglelefteq G_n = G.$$

Let $H \leq G$. Consider $H \cap G_i$ and $H \cap G_{i+1}$. Each is clearly a group. If $g \in H \cap G_i$ and $h \in H \cap G_{i+1}$, then $hgh^{-1} \in H$ since $g, h \in H$. Also, $hgh^{-1} \in G_i$, since $G_i \trianglelefteq G_{i+1}$, so we can conclude that

$$H \cap G_i \trianglelefteq H \cap G_{i+1}.$$

This also implies that $H \cap G_{i+1} \leq N_G(H \cap G_i)$. Hence, we see that H is solvable, as we can construct the series

$$1 \trianglelefteq H \cap G_1 \trianglelefteq \ldots H \cap G_{n-1} \trianglelefteq H \cap G_n = H \cap G = H.$$

□

3.5.4

Show that $S_n = \langle (12), (123 \ldots n) \rangle$ for all $n \geq 2$.

Proof. We want to show that for any transposition (pq) where $p < q$, $(pq) \in \langle (12), (123 \ldots n) \rangle$. Now,

$$(12 \ldots n)(12)(12 \ldots n)^{-1} = (12 \ldots n)(12)(1n \ldots 2) = (23)$$

and, in general,

$$(12 \ldots n)(m(m+1))(12 \ldots n)^{-1} = ((m+1)(m+2)).$$

Furthermore, for a transposition (pq),

$$(pq) = ((q-1)q) \ldots ((p+1)(p+2))(p(p+1))((p+1)(p+2)) \ldots ((q-1)q).$$

Since each term on the right is generated by (12) and $(12 \ldots n)$, then so is (pq). Since our choice of transposition was arbitrary, we see that every transposition in S_n is in $\langle (12), (12 \ldots n) \rangle$. Since, as we've seen, every element of S_n can be written as a product of transpositions, we see that, for all $\sigma \in S_n$, $\sigma \in \langle (12), (12 \ldots n) \rangle$. Hence, $S_n = \langle (12), (12 \ldots n) \rangle$. □

3.5.10

Find a composition series for A_4. Deduce that A_4 is solvable.
Answer:

$$1 \trianglelefteq \langle (12)(34) \rangle \trianglelefteq \langle (12)(34), (13)(24) \rangle \trianglelefteq A_4$$

is a composition series of A_4. To see that A_4 is, in fact, solvable, it suffices to note that

$$\langle (12)(34) \rangle / 1 \simeq \mathbb{Z}/2\mathbb{Z}$$

$$\langle (12)(34), (13)(24) \rangle / \langle (12)(34) \rangle \simeq \mathbb{Z}/2\mathbb{Z}$$

and

$$A_4 / \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}/3\mathbb{Z},$$

each of which is a simple abelian group.

♣

1

Let G be a group. The opposite group, G^{op}, is the group which is equal to G as a set, whose group law μ' is defined by

$$\mu'(x, y) = \mu_G(y, x) \quad \forall x, y \in G^{\text{op}}.$$

Prove that G^{op} is isomorphic to G.

Proof. Define $\phi : G \to G^{\text{op}}$ by

$$\phi(x) = x^{-1}.$$

This is well-defined since $x^{-1} \in G$ is equal to $x^{-1} \in G^{\text{op}}$. Then

$$ker(\phi)\{x \in G | x^{-1} = 1\} = \{1\}$$

so ϕ is injective. Also, for any $x \in G^{\text{op}}$,

$$\phi(x^{-1}) = (x^{-1})^{-1} = x,$$

so ϕ is surjective. Finally, for $x, y \in G$,

$$\phi(\mu(x, y)) = \mu(x, y)^{-1} = \mu(y^{-1}, x^{-1}) = \mu'(x^{-1}, y^{-1}) = \mu'(\phi(x), \phi(y)),$$

so ϕ is a homomorphism. Since ϕ is a bijective homomorphism, it is an isomorphism. □

2

Two homomorphisms f_1, f_2 from a group G_1 to a group G_2 are *conjugate* if there exists an element $g \in G_2$ such that $f_1(x) = gf_2(x)g^{-1}$ for all $x \in G_1$.

(a) Find all homomorphisms from S_3 to $\mathbb{C}^\times$.

Let $f : S_3 \to \mathbb{C}^\times$ be a homomorphism. Then $ker(f) \trianglelefteq S_3$. We know, from the last homework, that $\{1\}, \langle(12)\rangle, \langle(13)\rangle, \langle(23)\rangle, \langle(123)\rangle, S_3$ comprises the entire list of subgroups of S_3. If $ker(f) = \{1\}$, then f is a monomorphism, meaning $S_3 \simeq f(S_3)$. However, S_3 is not abelian, whereas $\mathbb{C}^\times$ is, so this is impossible. If $ker(f) = S_3$, then f is just the trivial homomorphism.

Now, we know that f induces an isomorphism $f' : S_3/ker(f) \to f(S_3)$. If $|ker(f)| = 2$, then

$$S_3/ker(f) \simeq \mathbb{Z}/3\mathbb{Z},$$

meaning $f(S_3)$ is a cyclic subgroup of $\mathbb{C}^\times$ of order 3. The only such group is the group of 3rd roots of unity, $\{1, e^{\frac{2\pi\sqrt{-1}}{3}}, e^{\frac{4\pi\sqrt{-1}}{3}}\}$. Hence, there are precisely two possibilities for f' given $ker(f)$. For example, if $ker(f) = \langle(12)\rangle$, then

$$S_3/ker(f) = S_3/\langle(12)\rangle = \{\langle(12)\rangle, (13)\langle(12)\rangle, (23)\langle(12)\rangle\}.$$

Then $f'(\langle(12)\rangle) = 1$, $f'((13)\langle(12)\rangle) = e^{\frac{2\pi\sqrt{-1}}{3}}$ or $f'((13)\langle(12)\rangle) = e^{\frac{4\pi\sqrt{-1}}{3}}$ and $f'((23)\langle(12)\rangle)$ is whatever remains. Hence, we deduce that

$$f(1) = f((12)) = 1, f((13)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}}$$

or

$$f(1) = f((12)) = 1, f((13)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((23)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}.$$

Similarly, if $ker(f) = \langle(13)\rangle$, then

$$f(1) = f((13)) = 1, f((12)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((23)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}$$

or

$$f(1) = f((13)) = 1, f((12)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((23)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}$$

and if $ker(f) = \langle(23)\rangle$, then

$$f(1) = f((23)) = 1, f((12)) = f((123)) = e^{\frac{2\pi\sqrt{-1}}{3}}, f((13)) = f((132)) = e^{\frac{4\pi\sqrt{-1}}{3}}$$

or

$$f(1) = f((23)) = 1, f((12)) = f((123)) = e^{\frac{4\pi\sqrt{-1}}{3}}, f((13)) = f((132)) = e^{\frac{2\pi\sqrt{-1}}{3}}.$$

Finally, if $|ker(f)| = 3$, then $ker(f) = \langle(123)\rangle$, so

$$S_3/ker(f) \simeq \mathbb{Z}/2\mathbb{Z}.$$

That is to say that $f(S_3)$ is a cyclic group of order two in $\mathbb{C}^\times$. The only such possibility is the group $\{1, -1\}$. This means $f(\langle(123)\rangle) = 1$ and $f((12)\langle(123)\rangle) = -1$. Specifically,

$$f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = -1.$$

Therefore, the above constitute all possible homomorphisms from S_3 to $\mathbb{C}^\times$.

♣

(b) Determine all homomorphisms from S_3 to S_3 up to conjugation.

Clearly the trivial homomorphism $f(x) = 1$ for all $x \in S_3$ is one such. Replacing $e^{\frac{2\pi\sqrt{-1}}{3}}$ with (123) and $e^{\frac{4\pi\sqrt{-1}}{3}}$ with (132) makes it clear that the only such homomorphisms having $\langle(12)\rangle$ as their kernel are f and g such that:

$$f(1) = f((12)) = 1, f((13)) = f((123)) = (123), f((23)) = f((132)) = (132)$$

and

$$g(1) = g((12)) = 1, g((13)) = g((123)) = (132), g((23)) = g((132)) = (123).$$

However, $f(x) = (12)g(x)(12)$ for all $x \in S_3$, so these two homomorphisms are conjugate. A similar argument gives a single homomorphism (up to conjugation) for each kernel $\langle(13)\rangle$ and $\langle(23)\rangle$. However, homomorphisms with different kernels of degree two will not be conjugate, as can be sen simply by noting that there is no $x \in S_3$ such that $x(123)x^{-1} = 1$.

Now, if f is a homomorphism with kernel $\langle(123)\rangle$, then, paralleling our arguments in the previous part, we see that either

$$f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (12)$$

or

$$f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (13)$$

or

$$f(1) = f((123)) = f((132)) = 1, f((12)) = f((13)) = f((23)) = (23).$$

If we call the first of these possibilities f_1, the second f_2 and the third f_3, then it is readily apparent that $f_1(x) = (23)f_2(x)(23)$, $f_1(x) = (13)f_3(x)(13)$ and $f_2(x) = (12)f_3(x)(12)$ for all $x \in S_3$, so f_1, f_2 and f_3 are conjugate.

Finally, if the kernel of a homomorphism from S_3 to S_3 is trivial, then that homomorphism is, in fact, an automorphism. In the last homework, we saw that any automorphism f of S_3 is of the form

$$f(x) = axa^{-1}$$

for some $a \in S_3$ and for all $x \in S_3$. Hence, if f and g are two automorphisms of S_3, then $f(x) = axa^{-1}$ and $g(x) = bxb^{-1}$ for some $a, b \in S_3$. However,

$$g(x) = bxb^{-1} = b(a^{-1}a)x(a^{-1}a)b^{-1} = (ba^{-1})(axa^{-1})(ab^{-1}) = (ba^{-1})f(x)(ba^{-1})^{-1},$$

so f and g are conjugate. Therefore, we conclude that, up to conjugation, there is exactly one homomorphism of S_3 into itself for each of the 6 possible kernels.

♣

(c) Prove that any two injective homomorphisms from S_3 to $GL_2(\mathbb{R})$ are conjugate.

Proof. Let f and g be monomorphisms from S_3 to $GL_2(\mathbb{R})$. Then $f(S_3)$ and $g(S_3)$ are isomorphic to S_3. Consider the map $g^{-1} \circ f : S_3 \to S_3$. Note that g^{-1} is an isomorphism. If $x, y \in S_3$ such that $(g^{-1} \circ f)(x) = (g^{-1} \circ f)(y)$, then

$$g^{-1}(f(x)) = g^{-1}(f(y))$$

so $f(x) = f(y)$, meaning $x = y$, so $g^{-1} \circ f$ is injective. Since S_3 is finite, $g^{-1} \circ f$ is clearly surjective. Also, if $x, y \in S_3$,

$$(g^{-1}\circ f)(xy) = g^{-1}(f(xy)) = g^{-1}(f(x)f(y)) = g^{-1}(f(x))g^{-1}(f(y)) = (g^{-1}\circ f)(x)(g^{-1}\circ f)(y),$$

so $g^{-1} \circ f$ is an automorphism. Hence, as shown in last week's homework, there exists $\tau \in S_3$ such that

$$(g^{-1} \circ f)(x) = \tau x \tau^{-1}$$

for all $x \in S_3$. Now,

$$f(x) = ((g\circ g^{-1})\circ f)(x) = (g\circ(g^{-1}\circ f))(x) = g((g^{-1}\circ f)(x)) = g(\tau x \tau^{-1}) = g(\tau)g(x)g(\tau^{-1}),$$

so we see that f and g are conjugate. Since our choice of f and g was arbitrary, we conclude that any two monomorphisms from S_3 to $GL_2(\mathbb{R})$ are conjugate. □

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