Solution Outlines for Chapter 20

 # 1: Describe the elements of $\mathbb{Q}(\sqrt[3]{5})$.

$\mathbb{Q}(5^{\frac{1}{3}}) = \{a + b5^{\frac{1}{3}} + c5^{\frac{2}{3}} | a, b, c \in \mathbb{Q}\}$

 # 3: Find the splitting field of $x^3 - 1$ over $\mathbb{Q}$. Express your answer in the form $\mathbb{Q}(a)$.

First, notice that $x^3 - 1 = (x - 1)(x^2 + x + 1)$ (see cyclotomic polynomials from class). So the roots of $x^3 - 1$ are 1 and $(-1 \pm \sqrt{-3})/2)$. Notice that 1 is already in $\mathbb{Q}$. By arguments like those in class, we see that we get $-(1 \pm \sqrt{-3})/2$ by simply adjoining $\sqrt{-3}$. So $\mathbb{Q}(\sqrt{-3})$ is the splitting field.

 # 6: Let $a, b \in \mathbb{R}$ with $b \neq 0$. Show that $\mathbb{R}(a + bi) = \mathbb{C}$.

It is clear that $\mathbb{R}(a + bi) \subseteq \mathbb{C}$. Now to see that $\mathbb{C} \subseteq \mathbb{R}(a + bi)$, I only need to show that $i \in \mathbb{R}(a + bi)$ (note: it is clear that $\mathbb{R}$ is there so getting i gives us all of $\mathbb{C}$). We see that $i = b^{-1}(a + bi) - b^{-1}a$ which is a real number times $(a + bi)$ plus a real number and so it is an element of $\mathbb{R}(a + bi)$. Thus we have equality.

 # 7: Find a polynomial $p(x)$ in $\mathbb{Q}[x]$ such that $\mathbb{Q}(\sqrt{1 + \sqrt{5}})$ is ring isomorphic to $\mathbb{Q}[x]/ < p(x) >$.

I need a polynomial so that it has rational coefficients, it is irreducible over $\mathbb{Q}$ and root $\sqrt{1 + \sqrt{5}}$ (Technically, we are using Theorem 20.3 that we haven't done yet). So I want $f(x)$ such that $f(x) = 0$ gives $x = \sqrt{1 + \sqrt{5}}$. This implies that $x^2 = 1 + \sqrt{5}$, or $x^2 - 1 = \sqrt{5}$. So $x^4 - 2x^2 + 1 = 5$ or $x^4 - 2x^2 - 4 = 0$. Let $p(x) = x^4 - 2x^2 - 4$. Then $p(x)$ is in $\mathbb{Q}[x]$ and has the needed root. Additionally, it is irreducible over $\mathbb{Q}$ (simply check in $\mathbb{Z}_3$).

 # 20: Let F be a field, and let a and b belong to F with $a \neq 0$. If c belongs to some extension of F, prove that $F(c) = F(ac + b)$. (F "absorbs" its own elements.)

This is akin to exercise 6. First, it is clear that $ac + b \in F(c)$ so $F(ac + b) \subseteq F(c)$. Now, $c = a^{-1}(ac + b) - a^{-1}b$ so $F(c) \subseteq F(ac + b)$ and we are done.

 # 21: Let $f(x) \in F[x]$ and let $a \in F$. Show that $f(x)$ and $f(x + a)$ have the same splitting field over F.

Suppose that the zeros of $f(x)$ are a_1, a_2, … a_k. Then the roots of $f(x + a)$ are $a_1 - a$, …, $a_k - a$. So by application of exercise 20 (up to k times), the splitting field $F(a_1, a_2, \ldots, a_k) = F(a_1 - a, \ldots a_k - a)$ and the splitting fields are the same.

 # 23: Determine all of the subfields of $\mathbb{Q}(\sqrt{2})$.

First notice that $\mathbb{Q}$ and $\mathbb{Q}(\sqrt{2})$ are subfields. Now suppose we have another subfield, K. K must contain $\mathbb{Q}$ and some element $a + b\sqrt{2}$ where $b \neq 0$. Now, K must contain $\mathbb{Q}(a + b\sqrt{2})$

but by exercise 20 this is just $\mathbb{Q}(\sqrt{2})$. So those are the only two subfields.

 # 27: Prove or disprove that $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{-3})$ are ring isomorphic.

Suppose that there is some map ϕ that is a ring isomorphism from $\mathbb{Q}(\sqrt{-3})$ to $\mathbb{Q}(\sqrt{3})$. Then $\phi(1) = 1$ so $\phi(-3) = -3$ (as we've seen before). So $-3 = \phi(-3) = \phi(\sqrt{-3}\sqrt{-3}) = (\phi(\sqrt{-3}))^2$. But this is a contradiction since $\phi(\sqrt{-3})$ is a real number. Hence, no such isomorphism exists.

 # 38: Show that $\mathbb{Q}(\sqrt{7}, i)$ is the splitting field for $x^4 - 6x^2 - 7$ (over $\mathbb{Q}$).

First we need to factor $x^4 - 6x^2 - 7$. We see that it factors to $(x^2 - 7)(x^2 + 1)$. Hence the roots are $\pm\sqrt{7}$ and $\pm i$. Notice that adjoining $\sqrt{7}$ gives us $-\sqrt{7}$ for free. Similarly, adjoining i also gives us $-i$. So the splitting field is $\mathbb{Q}(\sqrt{7}, i)$.