NOTES ON CONVERGENT SEQUENCES AND ON SUBSEQUENCES

Definition 2.7. A **sequence** in a set X is a function $\mathbf{P} : \mathbb{N} \to X$. We denote it by $\{\mathbf{P}(n)\}$ or, more commonly, by $\{p_n\}$.

Let $\{p_n\}_{n=1}^{\infty}$ be a sequence in a metric space. A **subsequence** of $\{p_n\}_{n=1}^{\infty}$ is a sequence of the form $\left\{p_{n(i)}\right\}_{i=1}^{\infty}$, where $n(1) < n(2) < n(3) < \cdots$. We typically write $\{p_{n_i}\}_{i=1}^{\infty}$, that is, we replace $n(i)$ by the notation n_i.

Remark. In $\{p_n\}$, n is represents a positive integer, i.e., $n = 1, 2, 3, \ldots$. Whereas, in $\left\{p_{n(i)}\right\}$, n represents an increasing function $n : \mathbb{N} \to \mathbb{N}$ and $i = 1, 2, 3, \ldots$.

Exercise. Convince yourself that a subsequence of a sequence $\mathbf{P} : \mathbb{N} \to X$ is simply a function $\mathbf{P} \circ n : \mathbb{N} \to X$, where $n : \mathbb{N} \to \mathbb{N}$ is increasing. **Hint**: The subsequence is $\{\mathbf{P} \circ n(i)\}$, which may be rewritten as $\left\{p_{n(i)}\right\}$.

Definition 3.1. A sequence $\{p_n\}$ in X is said to **converge** to $p \in X$ if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$.

Example. Consider the sequence $\{p_n\}_{n=1}^{\infty}$, where $p_n = (-1)^n$. Then $\{p_{2i}\}_{i=1}^{\infty}$ is a subsequence (where we define $n(i) = 2i$). Here, the original sequence is $\{-1, 1, -1, 1, \ldots\}$ and the subsequence is $\{1, 1, 1, 1, \ldots\}$. The original sequence diverges, whereas the subsequence converges.

Lemma. Suppose that $\{a_n\}$ is a sequence of positive numbers with $\lim_{n\to\infty} a_n = 0$. If $p \in X$ and $\{p_n\}$ is a sequence in X such that $d(p_n, p) \leq a_n$ for each $n \in \mathbb{N}$, then $\{p_n\}$ converges to p.

Proof. Let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$, there exists $N \in \mathbb{N}$ such that $a_n < \varepsilon$ for $n \geq N$. This implies that

$$d(p_n, p) \leq a_n < \varepsilon$$

for $n \geq N$. $\square$

Theorem 3.7. *The set of all subsequential limits of a sequence $\{p_n\}$ in a metric space X is a closed subset of X.*

Proof. Let E be the set of all $p \in X$ with the property that there exists a subsequence of $\{p_n\}$ which converges to p. Let $q \in E'$. We need to show that $q \in E$.

Let $n_1 = 1$. Let $k \geq 2$ and suppose that we have chosen positive integers $n_1 < n_2 < \cdots < n_{k-1}$. Since $q \in E'$, there exists $q_k \in E$ such that $d(q_k, q) < \frac{1}{k}$. Since $q_k \in E$, there exists a subsequence of $\{p_n\}$ which converges to q_k. This implies there exists an integer $n_k > n_{k-1}$ such that $d(p_{n_k}, q_k) < \frac{1}{k}$. Then

$$d(p_{n_k}, q) \leq d(p_{n_k}, q_k) + d(q_k, q) < \frac{2}{k}.$$

(We have defined the increasing sequence of positive integers $\{n_k\}_{k=1}^{\infty}$ by induction.) Since the subsequence $\{p_{n_k}\}_{k=1}^{\infty}$ has the property that $d(p_{n_k}, q) < \frac{2}{k}$ for $k \geq 2$ and since $\lim_{k\to\infty} \frac{2}{k} = 0$, we conclude by the lemma that $\{p_{n_k}\}_{k=1}^{\infty}$ converges to q. Hence $q \in E$. $\square$