Masters and Ph.D. Qualifying Exam **Analysis: Math 825/826**

Thursday, May 31, 2018, 11:30 am – 4:30 pm.

• Work 5 out of 6 problems. • Each problem is worth 20 points. • Write on one side of the paper only and hand your work in order.
• Do not interpret a problem in such a way that it becomes trivial.

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(1) Determine whether the following sequences converge and carefully justify your claims:

(i) $x_n = \dfrac{2n \cdot n!}{n^n}$; (ii) $y_n = \displaystyle\sum_{k=1}^{n} \frac{\cos(k!)}{k(k+1)}$.

(2) Find the domain of convergence and the sum of the series

$$\sum_{n\geq 0}(-1)^n \frac{x^{2n+1}}{2n+1}.$$

Show how one may use the sum of the series to provide an approximation for π up to three decimals. Be sure to provide all technical details.

(3) Define $f, \alpha \in \mathcal{B}([-2,2])$ by

$$f(x) := \begin{cases} -1, & x \in [-2,0); \\ 3, & x \in [0,2] \end{cases} \quad \text{and} \quad \alpha(x) := \begin{cases} -2, & x \in [-2,0]; \\ 1, & x \in (0,2]. \end{cases}$$

Determine whether f is Riemann-Stieltjes integrable with respect to α over $[-1,1]$. If it is, evaluate $\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}\alpha(x)$.

(4) (a) Given a set S, show that the function $\rho_\infty : \mathcal{B}(S) \times \mathcal{B}(S) \to \mathbb{R}$ defined by

$$\rho_\infty(f,g) := \operatorname{lub}\left(\{|f(x) - g(x)| : x \in S\}\right)$$

is a metric on $\mathcal{B}(S)$.

(b) Let $M > 0$ be given. Set

$$S := \left\{f \in \mathcal{C}_\mathrm{b}([0,1]) : f(0) = 0,\ f \text{ is differentiable on } (0,1), \text{ and } |f'(x)| \leq M \text{ for each } x \in (0,1)\right\}.$$

Determine whether the set S is compact in $(\mathcal{C}_\mathrm{b}([0,1]), \rho_\infty)$.

(5) (a) If $\{f_n\}_{n\geq 1}$ is an equicontinuous sequence of functions on a compact interval and $f_n \to f$ pointwise, prove that the convergence is uniform.

(b) Let $\alpha, M > 0$ be given, and suppose that $\{f_n\}_{n\geq 1}$ satisfies $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$, for all $n \geq 1$ and all x, y in an interval $[a,b]$. Show that $\{f_n\}_{n\geq 1}$ is equicontinuous.

(6) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and for all $n \geq 1$, put $f_n(x) = f(x + \frac{1}{n})$.

(a) Show that f_n converges uniformly, as $n \to \infty$, over any closed interval $[a,b]$.

(b) Give an example of a continuous function f for which the convergence is not uniform on $\mathbb{R}$.