Masters & Ph.D. Qualifying Exam **Analysis: Math 825/826**

May 26, 2016, 12:00-6:00p.m., Avery Hall 347

• Work 5 questions out 6. • Each problem is worth 20 points.
• Parts of each problem don't necessarily carry the same weight, and they can be unrelated. • Do not interpret a problem in such a way that it becomes trivial. • Write on one side of the paper only and hand your work in order.

(1) Consider the function: $f(x) = \dfrac{x}{1-x^2}$, $x \in (0,1)$.
(a) By using the ϵ–δ definition of the limit only, prove that f is continuous on $(0,1)$. (Note: You are not allowed to trivialize the problem by using properties of limits).
(b) Is f uniformly continuous on $(0,1)$? Justify your answer.

(2) Let $\{a_k\}_{k=1}^{\infty}$ be a bounded sequence of real numbers and E be given by:

$$E := \Big\{ s \in \mathbb{R} : \text{ the set } \{k \in \mathbb{N} : \ a_k \geq s\} \text{ has at most finitely many elements} \Big\}.$$

Prove that $\limsup\limits_{k\to\infty} a_k = \inf E$.

(3) Assume (X,d) is a compact metric space.
(a) Prove that X is both complete and separable.
(b) Suppose $\{x_k\}_{k=1}^{\infty} \subset X$ is a sequence such that the series $\sum_{k=1}^{\infty} d(x_k, x_{k+1})$ is convergent. Prove that the sequence $\{x_k\}_{k=1}^{\infty}$ converges in X.

(4) Suppose that $f : [0,2] \to \mathbb{R}$ is continuous on $[0,2]$, differentiable on $(0,2)$, and such that $f(0) = f(2) = 0$, $f(c) = 1$, for some $c \in (0,2)$. Prove that there is an $x \in (0,2)$ such that $|f'(x)| > 1$.

(5) Let $f_n(x) = n^{\beta} x\,(1-x^2)^n$, $x \in [0,1]$, $n \in \mathbb{N}$.
(a) Prove that $\{f_n\}_{n=1}^{\infty}$ converges point-wise on $[0,1]$ for every $\beta \in \mathbb{R}$.
(b) Show that the convergence in part (a) is uniform on $[0,1]$ for all $\beta < \frac{1}{2}$, but not uniform for any $\beta \geq \frac{1}{2}$.

(6) (a) Suppose $f : [-1,1] \longrightarrow \mathbb{R}$ is a bounded function that is continuous at 0. Let $\alpha(x) = -1$ for $x \in [-1,0]$ and $\alpha(x) = 1$ for $x \in (0,1]$. Prove that $f \in \mathcal{R}(\alpha)[-1,1]$, i.e., f is Riemann integrable with respect to α on $[-1,1]$, and $\int_{-1}^{1} f d\alpha = 2f(0)$.
(b) Let $g : [0,1] \longrightarrow \mathbb{R}$ be a continuous function such that $\int_0^1 g(x) x^{3k+2}\, dx = 0$, for all $k = 0, 1, 2, \cdots$. Prove that $g(x) = 0$ for all $x \in [0,1]$.