Day 4: Continuity of functions

Proposition 3.1 ([KRD10, Thm. 5.3.1]). *For a function $f : E \subset \mathbb{R}^n \to \mathbb{R}^m$ the following are equivalent:*

1) *f is continuous on E*
2) *For any $x \in E$ and $\epsilon > 0$ there exists some $\delta > 0$ such that $||f(y) - f(x)|| < \epsilon$ for all $y \in E$ with $||x - y|| < \delta$.*
3) *For every convergent sequence $\{x_n\} \subset E$ with $\lim_{n\to\infty} x_n = x \in E$, $\lim_{n\to\infty} f(x_n) = f(x)$.*
4) *For every open set $G \subset \mathbb{R}^m$ the set $f^{-1}(G)$ is open in E (with the subspace topology).*

Theorem 3.2 (Extreme Value Theorem / c.f. [Rud76, Thms. 4.14,15]). *If $f : K \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous and K is compact, then $f(K)$ is compact. In particular, given any continuous function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ there exist points $p, q \in [a, b]$ so that $f(p) = \sup_{[a,b]} f(x)$ and $f(q) = \inf_{[a,b]} f(x)$.*

Theorem 3.3 (c.f. [Rud76, Thm. 4.19]). *If $f : K \subset \mathbb{R}^n \to \mathbb{R}^m$ is continuous and K is compact, then f is uniformly continuous on K.*

Theorem 3.4 (Intermediate Value Theorem / c.f. [Rud76, Thm. 4.23]). *If $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is continuous and $f(a) < f(b)$, then for any real number x such that $f(a) < x < f(b)$ there exists some $c \in (a, b)$ such that $f(c) = x$.*

Theorem 3.5 (c.f. [Rud76, Thms. 4.29,30]). *If $f : (a, b) \to \mathbb{R}$ is monotonic then f is discontinuous on a set $E \subset (a, b)$ which is at most countable. Further, each point where f is discontinuous is a jump discontinuity in that the limits $f(x_i-)$ and $f(x_i+)$ both exist for each $x_i \in E$.*

In showing continuity directly from the ϵ-δ definition it is often helpful to recall the factorizations

$$x^2 - y^2 = (x - y)(x + y)$$
$$x^3 - y^3 = (x - y)(x^2 + xy + y^2).$$

Often, these will need to be manipulated to fit the problem at hand.

Warm-up problems.

1) State the definition of uniform continuity. Give an example of a function $f : (0, 1) \to \mathbb{R}$ which is continuous but not uniformly continuous.
2) (c.f. June 2012 #1b) If $f : [0, 1] \to [0, 1]$ is continuous, show that $f(x) = x$ for some $x \in [0, 1]$.
3) Give an ϵ-δ proof that $f(x) = x^{1/3}$ is continuous on $[0, 1]$.
(You may also wish to try this for $f(x) = \sqrt{x}$ and/or on the interval $[0, \infty)$ as in June 2011 and others.)
4) (June 2013 #1b) Prove that $\lim_{x\to-\infty} \frac{x-1}{x} = 1$.
5) A useful lemma: Let $\{x_n\}$ be a real sequence with $\liminf_{n\to\infty} x_n, \limsup_{n\to\infty} x_n \in \mathbb{R}$. Show that for any $\epsilon > 0$ there exists some $N \in \mathbb{N}$ so that
$$\left(\liminf_{n\to\infty} x_n\right) - \epsilon < x_n < \left(\limsup_{n\to\infty} x_n\right) + \epsilon \quad \text{for all } n \geq N.$$
What does this result imply if $\limsup_{n\to\infty} x_n < L$?

Problems.

6) (January 2009 #2a, obfuscated) Is $f(x) = \ln x$ uniformly continuous on $(0, 1]$? Prove your answer.
(You may also wish to try the same question with $f(x) = \sin(1/x)$ on $(0, \pi]$.)

7) (c.f. January 2007 #3a, June 2010 #2a) Prove Theorem 3.2 for a function $f : K \subset \mathbb{R} \to \mathbb{R}$. (The proof is nearly unchanged when n or m is greater than 1.)

8) (June 2009 #1) Give an ϵ-δ proof that $f(x) = \dfrac{x^2}{1 - x^2}$ is continuous on $(0, 1)$. Is f uniformly continuous on $(0, 1)$? Prove your answer.

9) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous and define $M : [a, b] \to \mathbb{R}$ by
$$M(x) = \sup\{f(y) : a \leq y \leq x\}.$$
Show that M is continuous on $[a, b]$.

10) ([KRD10, 5.6.H]) Show that a continuous function $f : \mathbb{R} \to \mathbb{R}$ cannot take on every real value exactly twice.

11) (June 2013 #5b) Let (X, d) be a complete metric space, $A \subset X$ be a bounded set, and $F : X \to X$. Assume there exists some $k > 0$ such that $d(F(a), F(b)) \leq kd(a, b)$ for all $a, b \in X$ and that $A \subset F(A)$. Provide a description of A if $k < 1$. Can anything be said about A if $k \geq 1$?

More problems.

12) (January 2012 #1b) Let $y \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be given. Suppose that for every sequence $\{x_n\}$ we have $\liminf_{n\to\infty} |f(x_n) - f(y)| \leq \liminf_{n\to\infty} |x_n - y|$. Prove that f is continuous at y.

13) Prove Theorem 3.3.

14) Prove Theorem 3.4.

References

[KRD10] Allan P. Donsig Kenneth R. Davidson. *Real analysis and applications*. Springer, 2010.
[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., USA, third edition, 1976.