Day 5: Differential calculus

Relevant information.

Theorem 4.1 ([Rud76, Thm. 5.8]). *If $f : [a, b] \to \mathbb{R}$ has a local maximum or minimum at $c \in (a, b)$ and $f'(c)$ exists, then $f'(c) = 0$.*

Theorem 4.2 (Mean value theorem / [Rud76, Thms. 5.9,10]). *If f is a real valued function which is continuous on $[a, b]$ and differentiable on (a, b), then there exists a point $\xi \in (a, b)$ so that*

$$\frac{f(b) - f(a)}{b - a} = f'(\xi).$$

Theorem 4.3 (Taylor's Theorem / [Rud76, Thm. 5.15], [KRD10, Thm. 10.1.3]). *Suppose $f : [a, b] \to \mathbb{R}$ is n times continuously differentiable on $[a, b]$, $f^{(n+1)}$ exists on (a, b), and $c \in [a, b]$. Then, for any $x \in [a, b]$ with $x \neq c$ there exists some ξ between c and x so that*

$$f(x) = \underbrace{f(c) + f'(c)(x - c) + \cdots + \frac{f^{(n)}(c)}{n!}(x - c)^n}_{P_n(x)} + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - c)^{n+1}.$$

In particular, if $|f^{(n+1)}(x)| \leq M$ on $[a, b]$ then

$$|f(x) - P_n(x)| \leq \frac{M|x - c|^{n+1}}{(n + 1)!}$$

for all $x \in [a, b]$.

Warm-up problems.

1) (June 1999 #10) Show that if f is differentiable on (a, b) with $f'(x) = 0$ on (a, b), then f is constant on (a, b).
2) ([Rud76, Exercise 5.1]) If $f : \mathbb{R} \to \mathbb{R}$ satisfies $|f(x) - f(y)| \leq (x - y)^2$ for all x, y, then f is constant.
3) ([KRD10, Exercise 6.2.B]) If $f : (a, b) \to \mathbb{R}$ is continuously differentiable and $f'(x_0) \neq 0$ for some $x_0 \in (a, b)$, then f is injective on some interval (c, d) containing x_0.

Problems.

4) (June 2005 #1a) Use the definition of the derivative to prove that if f and g are differentiable at x, then fg is differentiable at x.
5) (January 2006 #2b) Assume that f is differentiable at a. Evaluate

$$\lim_{x \to a} \frac{a^n f(x) - x^n f(a)}{x - a}, \quad n \in \mathbb{N}.$$

6) (June 2007 #3a) Suppose that $f, g : \mathbb{R} \to \mathbb{R}$ are differentiable, that $f(x) \leq g(x)$ for all $x \in \mathbb{R}$, and that $f(x_0) = g(x_0)$ for some x_0. Prove that $f'(x_0) = g'(x_0)$.
7) (June 2008 #3a) Prove that if f' exists and is bounded on $(a, b]$, then $\lim_{x \to a^+} f(x)$ exists.
8) (January 2012 #4b, extended) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f' \in C(\mathbb{R})$. Assume that there are $a, b \in \mathbb{R}$ with $\lim_{x \to \infty} f(x) = a$ and $\lim_{x \to \infty} f'(x) = b$. Prove that $b = 0$. Then, find a counterexample to show that the assumption $\lim_{x \to \infty} f'(x)$ exists is necessary.

9) (June 2012 #1a) Suppose that $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(0) = 0$. Prove that f is differentiable at $x = 0$ if and only if there is a function $g : \mathbb{R} \to \mathbb{R}$ which is continuous at $x = 0$ and satisfies $f(x) = xg(x)$ for all $x \in \mathbb{R}$.

More problems.

10) (January 2005 #6) Suppose that $f : [0, 1] \to \mathbb{R}$ is a differentiable function with $f(0) = 0$ and that there exists some $K > 0$ so that $|f'(x)| \leq K|f(x)|$ for all $x \in [0, 1]$. Prove that $f(x) = 0$ on $[0, 1]$.
Note: See [Rud76, Chap 5, #26] for a significant hint.

11) Assume that $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$ with $f(0) = f(1) = 0$ and $f(c) = 1$ for some $c \in (0, 1)$. Prove that there exists some $s \in (0, 1)$ such that $|f'(s)| > 2$.

12) (January 2010 #4) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$. If $\{x_n\}$ is an increasing sequence of real numbers converging to a and $\{y_n\}$ is a decreasing sequence of real numbers converging to a, prove that

$$\lim_{n \to \infty} \frac{f(y_n) - f(x_n)}{y_n - x_n} = f'(a).$$

13) Prove Theorem 4.1.

References

[KRD10] Allan P. Donsig Kenneth R. Davidson. *Real analysis and applications*. Springer, 2010.
[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., USA, third edition, 1976.