## DAY 6: INTEGRAL CALCULUS

**Relevant information.** For a bounded function $f : [a,b] \to \mathbb{R}$ and a monotonically increasing function $\alpha : [a,b] \to \mathbb{R}$ we say that $f$ is Riemann(-Steiltjes) integrable with respect to $\alpha$ and write $f \in \mathscr{R}(\alpha)$ on $[a,b]$ provided

$$\inf_{P\in\mathscr{P}[a,b]} U(P,f,\alpha) = \sup_{P\in\mathscr{P}[a,b]} L(P,f,\alpha).$$

The value of the integral is then this common quantity,

$$\int_a^b f d\alpha = \inf_{P\in\mathscr{P}[a,b]} U(P,f,\alpha) = \sup_{P\in\mathscr{P}[a,b]} L(P,f,\alpha).$$

Throughout, $\mathscr{P}[a,b]$ is the collection of all partitions of $[a,b]$.

**Theorem 5.1** (Riemann's condition / [Rud76, Thm. 6.6], [Apo74, Thm. 7.19]). *$f \in \mathscr{R}(\alpha)$ on $[a,b]$ if and only if for every $\epsilon > 0$ there exists a partition $P \in \mathscr{P}[a,b]$ such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

Frequently, we assume only that $\alpha$ is of bounded variation or even merely bounded. The following integration by parts formula is occasionally useful:

**Theorem 5.2** ([Apo74, Thm. 7.6]). *If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ then $\alpha \in \mathscr{R}(f)$ on $[a,b]$ and*

$$\int_a^b f d\alpha = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha df.$$

*Remark.* Some differences in the definition of $\mathscr{R}(\alpha)$ do exist between authors. Not all of these definitions are equivalent! This is unlikely to cause any issues when Riemann's condition is satisfied.

The ordinary Riemann integral is the case where $\alpha(x) = x$. In this instance, we write merely $f \in \mathscr{R}$ on $[a,b]$.

**Theorem 5.3** ([Rud76, Thm. 6.17]). *Assume $\alpha$ increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$ with $f : [a,b] \to \mathbb{R}$ bounded. Then, $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$ and, in that case,*

$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x)\,dx.$$

**Theorem 5.4** (First fundamental theorem of calculus / [Rud76, Thm. 6.20]). *If $f \in \mathscr{R}$ on $[a,b]$ and*

$$F(x) = \int_a^x f(t)dt,$$

*then $F$ is continuous on $[a,b]$ and differentiable at any $x_0 \in [a,b]$ where $f$ is continuous with $F'(x_0) = f(x_0)$.*

**Theorem 5.5** (Mean value theorem). *If $f : [a,b] \to \mathbb{R}$ is continuous then there exists some $c \in (a,b)$ such that*

$$\frac{1}{b-a}\int_a^b f(x)dx = f(c).$$

**Warm-up problems.**

1) ([KRD10, 6.4.N]) If $f$ and $g$ are bounded on $[a, b]$ and both are Riemann integrable on $[a, b]$, show that $fg \in \mathscr{R}$ on $[a, b]$.

2) ([Apo74, 7.12] c.f. [Rud76, p. 138 #3]) Give an example of a bounded function $f$ and an increasing function $\alpha$ defined on $[a, b]$ such that $|f| \in \mathscr{R}(\alpha)$ but $f \notin \mathscr{R}(\alpha)$.

3) (January 2007 #1) Let $f(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 0$. (a) Use an $\epsilon$-$\delta$ proof to show that $f$ is continuous on $(0, \infty)$. (b) Use an $\epsilon$-$\delta$ proof to show that $f$ is differentiable on $(0, \infty)$.

4) ([Apo74, 7.2]) If $f \in \mathscr{R}(\alpha)$ on $[a, b]$ and $\int_a^b f d\alpha = 0$ for every $f$ which is monotonic on $[a, b]$, prove that $\alpha$ must be constant on $[a, b]$.

**Problems.**

5) (January 2006 #4b) Suppose that $f$ is continuous and $f(x) \geq 0$ on $[0, 1]$. If $f(0) > 0$, prove that $\int_0^1 f(x)dx > 0$.

6) (June 2005 #1b) Use the definition of the Riemann integral to prove that if $f$ is bounded on $[a, b]$ and is continuous everywhere except for finitely many points in $(a, b)$, then $f \in \mathscr{R}$ on $[a, b]$.

7) (January 2010 #5) Suppose that $f : [a, b] \to \mathbb{R}$ is continuous, $f \geq 0$ on $[a, b]$, and put $M = \sup\{f(x) : x \in [a, b]\}$. Prove that

$$\lim_{p \to \infty} \left( \int_a^b f(x)^p \, dx \right)^{1/p} = M.$$

8) (January 2009 #4b) Let $f$ be a continuous real-valued function on $[0, 1]$. Prove that there exists at least one point $\xi \in [0, 1]$ such that $\int_0^1 x^4 f(x)\, dx = \frac{1}{5} f(\xi)$.

9) (June 2009 #5b) Let $\phi$ be a real-valued function defined on $[0, 1]$ such that $\phi$, $\phi'$, and $\phi''$ are continuous on $[0, 1]$. Prove that

$$\int_0^1 \cos x \frac{x\phi'(x) - \phi(x) + \phi(0)}{x^2}\, dx < \frac{3}{2} ||\phi''||_\infty,$$

where $||\phi''||_\infty = \sup_{[0,1]} |\phi''(x)|$. Note that 3/2 may not be the smallest possible constant.

10) (Essentialy June 2013 #7) Prove Theorem 5.3.

## References


[Apo74] Tom M. Apostol. *Mathematical Analysis*. Addison-Wesley, second edition, 1974.

[KRD10] Allan P. Donsig Kenneth R. Davidson. *Real analysis and applications*. Springer, 2010.

[Rud76] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill, Inc., USA, third edition, 1976.