Day 8: Miscellaneous Topics

Bounded Variation.

1) (January 2018) Let $f\colon [a,b] \to \mathbb{R}$. Suppose $f \in \mathrm{BV}[a,b]$. Prove f is the difference of two increasing functions.

2) (January 2007, 6a) Let f be a function of bounded variation on $[a,b]$. Furthermore, assume that for some $c > 0$, $|f(x)| \geq c$ on $[a,b]$. Show that $g(x) = 1/f(x)$ is of bounded variation on $[a,b]$.

3) (January 2017, 2a) Define $f\colon [0,1] \to [-1,1]$ by
$$f(x) := \begin{cases} x\sin\left(\frac{1}{x}\right) & 0 < x \leq 1 \\ 0 & x = 0 \end{cases}$$
Determine, with justification, whether f is if bounded variation on the interval $[0,1]$.

4) (January 2020, 6a) Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and a strictly increasing sequence $\{x_n\}_{n=1}^{\infty} \subseteq (0,1)$ be given. Assume that $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, and define $\alpha\colon [0,1] \to \mathbb{R}$ by
$$\alpha(x) := \begin{cases} a_n & x = x_n \\ 0 & \text{otherwise} \end{cases}.$$
Prove or disprove: α has bounded variation on $[0,1]$.

Metric Spaces and Topology.

1) Find an example of a metric space X and a subset $E \subseteq X$ such that E is closed and bounded but not compact.

2) (May 2017 6) Let (X,d) be a metric space. A function $f\colon X \to \mathbb{R}$ is said to be lower semi-continuous (l.s.c) if $f^{-1}(a,\infty) = \{x \in X : f(x) > a\}$ is open in X for every $a \in \mathbb{R}$. Analogously, f is upper semi-continuous (u.s.c) if $f^{-1}(-\infty,b) = \{x \in X : f(x) < b\}$ is open in X for every $b \in \mathbb{R}$.

   (a) Prove that a function $f\colon X \to \mathbb{R}$ is continuous if and only if f is both l.s.c. and u.s.c.

   (b) Prove that f is lower semi-continuous if and only if $\liminf_{n\to\infty} f(x_n) \geq f(x)$ whenever $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that $x_n \to x$ in X.

3) (January 2017 3) Let (X,d) be a compact metric space. Suppose that $f_n\colon X \to [0,\infty)$ is a sequence of continuous functions with $f_n(x) \geq f_{n+1}(x)$ for all $n \in \mathbb{N}$ and $x \in X$, and such that $f_n \to 0$ pointwise on X. Prove that $\{f_n\}_{n=1}^{\infty}$ converges uniformly on X.

Integral Calculus.

1) (June 2014 1)Define $\alpha\colon [-1,1] \to \mathbb{R}$ by
$$\alpha(x) := \begin{cases} -1 & x \in [-1,0] \\ 1 & x \in (0,1]. \end{cases}$$
Let $f\colon [-1,1] \to \mathbb{R}$ be a function that is uniformly bounded on $[-1,1]$ and continuous at $x = 0$, but not necessarily continuous for $x \neq 0$. Prove that f is Riemann-Stieltjes integrable

with respect to α over $[-1,1]$ and that

$$\int_{-1}^{1} f(x)d\alpha(x) = 2f(0).$$

2) (June 2017 2) Prove : $f \in \mathcal{R}(\alpha)$ on $[a,b]$ if and only if for any $a < c < b$, $f \in \mathcal{R}(\alpha)$ on $[a,c]$ and on $[c,b]$. In addition, if either condition holds, then we have that

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha.$$

3) (Spring 2017 7) Prove that if $f \in \mathcal{R}$ on $[a,b]$ and $\alpha \in C^1[a,b]$, then the Riemann integral $\int_a^b f(x)\alpha'(x)dx$ exists and

$$\int_a^b f(x)d\alpha(x) = \int_a^b f(x)\alpha'(x)dx.$$

Sequences and Series (and of Functions).

1) (January 2006 1) Let the power series series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ have radii of convergence R_1 and R_2, respectively.

   (a) If $R_1 \neq R_2$, prove that the radius of convergence, R, of the power series $\sum_{n=0}^{\infty}(a_n+b_n)x^n$ is $\min\{R_1, R_2\}$. What can be said about R when $R_1 = R_2$?

   (b) Prove that the radius of convergence, R, of $\sum_{n=0}^{\infty} a_n b_n x^n$ satisfies $R \geq R_1 R_2$. Show by means of example that this inequality can be strict.

2) Show that the infinite series $\sum_{n=0}^{\infty} x^n 2^{-nx}$ converges uniformly on $[0, B]$ for any $B > 0$. Does this series converge uniformly on $[0,\infty)$?

3) (January 2006 4a) Let

$$f_n(x) = \begin{cases} \frac{1}{n} & x \in (\frac{1}{2^{n+1}}, \frac{1}{2^n}] \\ 0 & \text{otherwise.} \end{cases}$$

   Show that $\sum_{n=1}^{\infty} f_n$ does not satisfy the Weierstrass M-test but that it nevertheless converges uniformly on $\mathbb{R}$.

4) Let $f_n\colon [0,1) \to \mathbb{R}$ be the function defined by

$$f_n(x) := \sum_{k=1}^{n} \frac{x^k}{1+x^k}.$$

   (a) Prove that f_n converges to a function $f\colon [0,1) \to \mathbb{R}$.

   (b) Prove that for every $0 < a < 1$ the convergence is uniform on $[0,a]$.

   (c) Prove that f is differentiable on $(0,1)$.