Topology Qual Workshop Day 9: Homology

Warm-up Problems:

- Compute the homology groups of S^n for all $n \geq 0$.
- Compute the homology of the torus, $T^2 = S^1 \times S^1$.

(1) (Michigan Sept '10) Consider the 2-dimensional torus T and the topological space

$$X = T \times [-1,1]/\sim$$

where $(x,t) \sim (x',t')$ if either $(x,t) = (x',t')$ or $t = t' \in \{-1,1\}$. Compute $H_*(X,\mathbb{Z})$.

(2) (Michigan Jan '10) Let $S^1 = \{z \in \mathbb{C} || |z|| = 1\}$. Let X be obtained from the space $S^1 \times [0,1]$ by identifying every point $(z,0)$ with the point $(z^4,1)$ with the quotient topology. Compute the fundamental group and homology of X.

(3) (Michigan May '09). Let $S^1 = \{(x,y,0,0) \in \mathbb{R}^4 | x^2 + y^2 = 1\}$ be the unit circle and consider $M = \mathbb{R}^4 \setminus S^1$. Compute the fundamental group $\pi_1(M)$ and the homology groups $H_*(M)$ of M.

(4) (Michigan Sept '08) Compute the homology of the space formed as the union of the unit sphere $\{(x,y,z) | x^2 + y^2 + z^2 = 1\}$ and the closed interval along the z-axis from $(0,0,-1)$ to $(0,0,1)$.

(5) (Arizona Aug '06) Compute the singular homology groups $H_*(X,\mathbb{Z})$ of the space $X = \mathbb{R}^3 \setminus A$, where A is a subset of $\mathbb{R}^3$ homeomorphic to the disjoint union of two unlinked circles.

(6) Use the Meyer-Vietoris sequence to calculate the homology of $X \vee Y$.

(7) (Jan '12) Let X be a space obtained by identifying the three vertices of a standard 2-simplex.

  (a) Describe a structure of a Δ-complex on X.

  (b) Write down the chain complex corresponding to the Δ-complex in (a)

  (c) Compute the homology of X.

(8) (May '13) Let Y be the standard 3-simplex Δ^3 with a total ordering on its four vertices. Let X be the Δ-complex obtained from Y by identifying, for each $k \leq 3$, all of its k-dimensional faces such that the identifications respect the vertex ordering. Thus X has a single k-simplex for each $k \leq 3$. Compute the simplicial homology groups of the Δ-complex X.