Partial Solutions to Folland's Real Analysis: Part I

(Assigned Problems from MAT1000: Real Analysis I)

Jonathan Mostovoy - 1002142665
University of Toronto

January 20, 2018

Contents

1 Chapter 1 **3**
1.1 Folland 1.2 3
1.2 Folland 1.4 3
1.3 Folland 1.5 4
1.4 Boxes vs cylinder sets w.r.t. σ-algebras 5
1.5 Folland 1.7 5
1.6 Folland 1.8 6
1.7 Folland 1.9 7
1.8 Folland 1.10 7
1.9 Folland 1.13 8
1.10 Folland 1.17 8
1.11 Folland 1.18 9
1.12 Folland 1.26 10
1.13 Folland 1.28 11
1.14 Folland 1.30 13
1.15 Folland 1.31 13
1.16 Folland 1.33 14

2 Chapter 2 **15**
2.1 Folland 2.1 15
2.2 Folland 2.2 15
2.3 Folland 2.3 17
2.4 Folland 2.4 17
2.5 Folland 2.7 17
2.6 Folland 2.8 18
2.7 Folland 2.9 18
2.8 Folland 2.10 20
2.9 Folland 2.12 21
2.10 Folland 2.13 21
2.11 Folland 2.14 22
2.12 Folland 2.16 23
2.13 Folland 2.17 24
2.14 Differentiable functions are Borel Measurable 24
2.15 Folland 2.20 25

2.16 Folland 2.21 . . . 25
2.17 Folland 2.24 . . . 26
2.18 Folland 2.34 . . . 26
2.18.1 Folland 2.33 . . . 26
2.19 Folland 2.39 . . . 27
2.20 Folland 2.42 . . . 27
2.21 Folland 2.44: Lusin's Theorem . . . 28
2.22 Folland 2.46 . . . 28
2.23 Folland 2.48 . . . 29
2.24 Folland 2.49 . . . 29

3 Chapter 3 **30**
3.1 Folland 3.2 . . . 30
3.2 Folland 3.7 . . . 31
3.3 Folland 3.12 . . . 32
3.4 Folland 3.13 . . . 33
3.5 Folland 3.17 . . . 33
3.6 Folland 3.20 . . . 34
3.7 Folland 3.21 . . . 35
3.8 Folland 3.24 . . . 36
3.9 Folland 3.25 . . . 37
3.10 Folland 3.26 . . . 38

4 Chapter 5 **39**
4.1 Folland 5.1 . . . 39
4.2 Folland 5.2 . . . 39
4.3 Folland 5.5 . . . 40
4.4 Folland 5.6 . . . 40
4.5 Folland 5.9 . . . 43

5 Chapter 6 **44**
5.1 Folland 6.3 . . . 44
5.2 Folland 6.4 . . . 45
5.3 Folland 6.5 . . . 46
5.4 Folland 6.7 . . . 47
5.5 Folland 6.10 . . . 48
5.6 Folland 6.14 . . . 49

1 Chapter 1

1.1 Folland 1.2

Prove the following Proposition:

Proposition. 1.1:

$\mathcal{B}_{\mathbb{R}}$ is generated by each of the following:

(a) *the open intervals:* $\mathcal{E}_1 = \{(a,b) \mid a < b\}$,

(b) *the closed intervals:* $\mathcal{E}_2 = \{[a,b] \mid a < b\}$,

(c) *the half-open intervals:* $\mathcal{E}_3 = \{(a,b] \mid a < b\}$ or $\mathcal{E}_4 = \{[a,b) \mid a < b\}$,

(d) *the open rays:* $\mathcal{E}_5 = \{(a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty,a) \mid a \in \mathbb{R}\}$,

(e) *the closed rays:* $\mathcal{E}_7 = \{[a,\infty) \mid a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty,a] \mid a \in \mathbb{R}\}$,

Proof. Most of the proof is already completed by Folland. What was shown is that $\mathcal{M}(\mathcal{E}_j) \subset \mathcal{B}_{\mathbb{R}}\ \forall j = 1,\ldots,8$. To finish the proof and show $\mathcal{B}_{\mathbb{R}} = \mathcal{M}(\mathcal{E}_j)\ \forall j$, we can simply show that $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_j)\ \forall j$. By invoking Lemma 1.1, if the family of open sets lie in $\mathcal{M}(\mathcal{E}_j)$, then it must be that $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}(\mathcal{E}_j)$. Furthermore, it is actually sufficient to only show that all the open intervals lie in $\mathcal{M}(\mathcal{E}_j)$ since every open set in $\mathbb{R}$ is a countable union of open intervals. Thus, we complete our proof by directly showing the following:

1. $(a,b) \in \mathcal{E}_1 \Rightarrow (a,b) \in \mathcal{M}(\mathcal{E}_2)$.
2. $(a,b) = \cup_1^\infty [a+n^{-1}, b-n^-1] \in \mathcal{M}(\mathcal{E}_2)$
3. $(a,b) = \cup_1^\infty (a, b-n^-1] \in \mathcal{M}(\mathcal{E}_3)$
4. $(a,b) = \cup_1^\infty [a+n^{-1}, b) \in \mathcal{M}(\mathcal{E}_4)$
5. $(a,b) = (a,\infty) \cap (-\infty,b) = (a,\infty) \cap [b,\infty)^c = (a,\infty) \cap \left(\cap_1^\infty (b-n^{-1},\infty)\right)^c \in \mathcal{M}(\mathcal{E}_5)$
6. $(a,b) = (a,\infty) \cap (-\infty,b) = (-\infty,a]^c \cap (-\infty,b) = \left(\cap_1^\infty (-\infty,a+n^{-1})\right)^c \cap (-\infty,b) \in \mathcal{M}(\mathcal{E}_6)$
7. $(a,b) = (a,\infty) \cap (-\infty,b) = \left(\cup_1^\infty [a+n^{-1},\infty)\right) \cap [b,\infty)^c \in \mathcal{M}(\mathcal{E}_7)$
8. $(a,b) = (a,\infty) \cap (-\infty,b) = (-\infty,a]^c \cap \left(\cup_1^\infty (-\infty,b-n^{-1}]\right) \in \mathcal{M}(\mathcal{E}_8)$

$\square$

1.2 Folland 1.4

Prove the following proposition:

Proposition. 1.2:

An algebra $\mathcal{A}$ is a σ-algebra $\iff$ $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \cdots$, then $\cup_1^\infty E_j \in \mathcal{A}$).

Proof. The forward direction (σ-algebra $\Rightarrow$ closed under countable increasing unions) is by the definition of σ-algebra (closed under countable unions). The backward direction (closed under countable increasing unions $\Rightarrow$ closed under countable increasing unions $\Rightarrow$ σ-algebra) is slightly more involved:

If $\{F_i\}_1^\infty \in \mathcal{A}$, then let us define $E_j := \cup_1^j F_i$. Since countable unions of countable unions is countable, and since $\{E_j\}_1^\infty$ has the property of $E_1 \subset E_2 \subset \cdots$, then we know that $\cup_1^\infty E_j \in \mathcal{A}$. However, since it is also the case that $\cup_1^\infty F_i = \cup_1^\infty E_j$, we can conclude that $\cup_1^\infty F_i \in \mathcal{A}$ as well, and thus proving the backward direction. $\square$

1.3 Folland 1.5

Prove the following Proposition:

Proposition. 1.3:

If $\mathcal{M}(\mathcal{E})$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}(\mathcal{E})$ is the union of the σ-algebras generated by $\mathcal{F}_\alpha$ as $\mathcal{F}_\alpha$ ranges over all countable subsets of $\mathcal{E}$.

Proof. We use the notation $\mathcal{F}_\alpha$ to denote a countable subset of $\mathcal{E}$, and we let $\mathcal{F} := \{\mathcal{F}_\alpha \mid \alpha \in A\}$ denote the (likely uncountable) set of all countable subsets of $\mathcal{E}$. Let us also define $\hat{\mathcal{M}} := \cup_{\alpha \in A}\mathcal{M}(\mathcal{F}_\alpha)$. We proceed now by first showing that $\hat{\mathcal{M}}$ is indeed a σ-algebra by showing that $\hat{\mathcal{M}}$ is closed under countable unions and compliments:

Suppose $\{E_i\}_1^\infty \in \hat{\mathcal{M}}$. Since $\hat{\mathcal{M}}$ is simply the union of a many σ-algebras, we know immediately that $\forall E_i$ $\exists$ at least one $\mathcal{F}_i$ s.t. $E_i \in \mathcal{M}(\mathcal{F}_i)$. Since a countable union of countable elements is countable, if we define $H := \cup_1^\infty \mathcal{F}_i$ where $E_i \in \mathcal{M}(\mathcal{F}_i)$, we know that H is also countable subset of $\mathcal{E}$. We can now look at the properties of the following σ-algebra: $\mathcal{M}(H)$.

(1) Since $\mathcal{F}_i \subset H \subset \mathcal{M}(H) \Rightarrow \mathcal{M}(\mathcal{F}_i) \subset \mathcal{M}(H)$ (by Lemma 1.1), and since $E_i \in \mathcal{M}(\mathcal{F}_i)$, we can say that $\{E_i\}_1^\infty \in \mathcal{M}(H)$.

(2) Since H is a countable subset of $\mathcal{E}$, we know that $\exists\beta$ s.t. $H = \mathcal{F}_\beta$, and hence $\mathcal{M}(H) \subset \hat{\mathcal{M}}$.

Therefore, since $\mathcal{M}(H)$ is by construction a σ-algebra and from (1) ($\{E_i\}_1^\infty \in \mathcal{M}(H)$) it $\Rightarrow \cup_1^\infty E_i \in \mathcal{M}(H)$, and by (2) ($\mathcal{M}(H) \subset \hat{\mathcal{M}}$) $\Rightarrow \cup_1^\infty E_i \in \hat{\mathcal{M}}$.

To now show $\hat{\mathcal{M}}$ is closed under compliments, suppose $E \in \hat{\mathcal{M}}$. By the same argument already used, there must exist a countable subset $\mathcal{F}_\alpha \subset \mathcal{E}$ s.t. $E \in \mathcal{M}(F_\alpha)$, and obviously since $\mathcal{M}(\mathcal{F}_\alpha)$ is a σ-algebra, $E^c \in \mathcal{M}(F_\alpha)$. Therefore, since $\mathcal{M}(\mathcal{F}_\alpha) \subset \hat{\mathcal{M}} \Rightarrow E^c \in \hat{\mathcal{M}}$. We have thus shown that $\hat{\mathcal{M}}$ is is closed under countable unions and compliments, and hence a σ-algebra.

To neatly finish up our proof, let us first note that $\forall\alpha \in A$, $\mathcal{F}_\alpha \subset \mathcal{E} \Rightarrow \mathcal{M}(\mathcal{F}_\alpha) \subset \mathcal{M}(\mathcal{E})$, and thus we can also say $\hat{\mathcal{M}} \subset \mathcal{M}(\mathcal{E})$. To show the opposite relation, let $\varepsilon \in \mathcal{E}$, then ε is trivially countable, so $\exists\beta$ s.t. $\varepsilon = \mathcal{F}_\beta \Rightarrow \varepsilon \in \hat{\mathcal{M}}$. Now since this is true $\forall\varepsilon \in \mathcal{E}$, we can say that $\mathcal{E} \subset \hat{\mathcal{M}}$, which therefore (again by Lemma 1.1) $\Rightarrow \mathcal{M}(\mathcal{E}) \subset \mathcal{M}$. By showing both opposite relations, we can thus conclude that $\mathcal{M}(\mathcal{E}) = \hat{\mathcal{M}}$. $\square$

1.4 Boxes vs cylinder sets w.r.t. σ-algebras

Exercise. 1.1:

Let A be an index set, $\{X_\alpha\}_{\alpha\in A}$ a family of non-empty sets and for each $\alpha \in A$, $\mathcal{M}_\alpha$ be a σ-algebra on X_α. Consider the product space:

$$X = \prod_{\alpha\in A} X_\alpha$$

Let $\mathcal{M}$ be the σ-algebra generated by the cylinder sets $\mathcal{C} := \{\pi_\alpha^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha, \alpha \in A\}$, and $\mathcal{M}^*$ be the one generated by boxes $\mathbb{B} := \{\prod_{\alpha\in A} E_\alpha \mid E_\alpha \in \mathcal{M}_\alpha\}$. Show that $\mathcal{M} \subset \mathcal{M}^*$, but in general $\mathcal{M} \neq \mathcal{M}^*$

Hint 1: Proposition 1.3 implies that if A is countable then $\mathcal{M} = \mathcal{M}^*$; we should thus take A to be not countable.)

Hint 2: You might find useful to first prove the following intermediate result. For any $A' \subset A$, let $\mathcal{M}_{A'} = \mathcal{M}(\{\pi^{-1}(E_\alpha) \mid E_\alpha \in \mathcal{M}_\alpha, \alpha \in A'\})$; let now

$$\tilde{\mathcal{M}} = \bigcup_{A'\subset A \text{ countable}} \mathcal{M}_{A'}$$

Then show that $\mathcal{M} = \tilde{\mathcal{M}}$. (Hint^2: show that $\tilde{\mathcal{M}}$ is a σ-algebra which contains the cylinders...) The above can be loosely stated as "any set in $\mathcal{M}$ is determined by countably many coordinates"

**Please note the notation used for the box and cylinder sets above.

Answer: To show $\mathcal{M} \subset \mathcal{M}^*$, note that $\pi_\alpha^{-1}(E_\alpha) = \prod_{\beta\in A} E_\beta$, where $E_\beta = X_\beta\ \forall\ \beta \neq \alpha$. In this form, it is clear that $\mathcal{C} \subset \mathbb{B} \subset \mathcal{M}^* \Rightarrow \mathcal{M} \subset \mathcal{M}^*$ (by Lemma 1.1).

Next, let us prove that $\mathcal{M} = \tilde{\mathcal{M}}$:

Proof. Suppose $\{F_i\}_1^\infty \in \tilde{\mathcal{M}}$. Then since $\tilde{\mathcal{M}}$ is a union of σ-algebras, it must be that $F_i \in \mathcal{M}_{A'}$ for at least one A'. Taking A'' to be the union of <u>one of</u> the A's which satisfies $F_i \in \mathcal{M}_{A'}$ for each i. Thus, A'' will naturally also be a countable set. Since A'' is a countable set, $\mathcal{M}_{A''} \subset \tilde{\mathcal{M}} \Rightarrow \cup_1^\infty F_i \in \tilde{\mathcal{M}}$ by Lemma 1.1.

Next, suppose $F \in \tilde{\mathcal{M}}$, then $\exists A'$ s.t. $F \in \mathcal{M}_{A'}$, which implies $F^c \in \mathcal{M}_{A'}$, and since $\mathcal{M}A' \subset \tilde{\mathcal{M}}$, $\Rightarrow F^c \in \tilde{\mathcal{M}}$, and hence $\tilde{\mathcal{M}}$ is indeed a σ-algebra.

Next, since $A' \subset A$, $\Rightarrow \mathcal{M}_{A'} \subset \mathcal{M}(= \mathcal{M}_A)\ \forall A'$, and thus since $\mathcal{M}_{A'} \subset \mathcal{M}\ \forall A' \Rightarrow \tilde{\mathcal{M}} \subset \mathcal{M}$. To show the opposite inclusion, we know that $\forall\ \alpha \in A\ \exists$ a countable subset $A' \subset A$ s.t. $\alpha \in A'$, namely $\{\alpha\}$. In this form, it is perfectly clear that $\pi^{-1}(E_\alpha) \subset \tilde{\mathcal{M}}$, since $\pi^{-1}(E_\alpha) \in \mathcal{M}_{A'=\{\alpha\}} \Rightarrow \mathcal{M} \subset \tilde{\mathcal{M}}$. And thus $\mathcal{M} = \tilde{\mathcal{M}}$. $\square$

Let us now turn our attention to the form in which the generating family of sets for $\mathcal{M}_{A'}$ takes. Each set is in the form $\pi^{-1}(E_\alpha) = (\prod_{\beta\in A'} E_\beta)\times(\prod_{\gamma\in A\setminus A'} X_\gamma)$, where A' is a countable set, and $E_\beta = X_\beta\ \forall\beta \neq \alpha$. In this form, it is clear that after countably many intersections, compliments and unions, $\forall E \in \mathcal{M}_{A'}$, E will still be in the form of $(\prod_{\beta\in A'} E_\beta) \times (\prod_{\gamma\in A\setminus A'} X_\gamma)$, where A' is a countable set, and $E_\beta \in \mathcal{M}_\beta$. However, when looking at the boxes, it is clear that $\exists E \in \mathcal{M}(\mathbb{B})$ s.t. $E = (\prod_{\beta\in B} E_\beta) \times (\prod_{\gamma\in A\setminus B} X_\gamma)$, where B is an uncountable set, and $E_\beta \in \mathcal{M}_\beta$.

1.5 Folland 1.7

Prove the following Proposition:

Proposition. 1.4:

If $\mu_1, \dots, \mu_n$ are measures on $(X, \mathcal{M})$ and $a_1, \dots, a_n \in [0, \infty)$, then $\sum_1^n a_j \mu_j$ is a measure on $(X, \mathcal{M})$.

Proof. Since μ_i $i \in \{1, \dots, n\}$ are measures, we know that $\mu_i(\varnothing) = 0$ $\forall i = 1, \dots, n$, and therefore $\mu := \sum_1^n a_j \mu_j(\varnothing) = 0$. Next, suppose $\{E_j\}_1^\infty \in \mathcal{M}$ and $\{E_j\}_1^\infty$ disjoint, then:

$$\mu\left(\bigsqcup_{i=1}^{n} E_i\right) = \sum_{j=1}^{n}\left(a_j \cdot \mu_j\left(\bigsqcup_{i=1}^{n} E_i\right)\right) = \sum_{j=1}^{n}\left(a_j \cdot \sum_{i=1}^{\infty} \mu_j\left(E_i\right)\right) = \sum_{j=1}^{n}\left(\sum_{i=1}^{\infty} a_j \cdot \mu_j\left(E_i\right)\right) = \sum_{i=1}^{\infty} \mu(E_i)$$

□

1.6 Folland 1.8

Prove the following Proposition:

Proposition. 1.5:

If $(X, \mathcal{M}, \mu)$ is a measure space and $\{E_j\}_1^\infty \subset \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\cup_1^\infty E_j) < \infty$.

Proof. We first recall the definitions of lim inf and lim sup for a sequence of sets as:

$$\liminf_{n\to\infty}(F_n) := \bigcup_{k=1}^{\infty}\left(\bigcap_{n=k}^{\infty} F_n\right), \quad \text{and} \quad \limsup_{n\to\infty}(F_n) := \bigcap_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} F_n\right)$$

We now quickly prove the following Lemma:

Lemma. 1.1: A Corollary of Monotonicity and Subadditivity - (Again!)

If $\{B_i\}_1^\infty \subset \mathcal{M}$, then:

(a) $\mu(\cap_1^\infty B_i) \leq \mu(B_1)$ (or $\mu(B_k)$ by switching B_1 for B_k).

(b) $\mu(\cup_1^\infty B_i) \geq \mu(B_1)$ (or $\mu(B_k)$ by switching B_1 for B_k).

Note: I sorta forget these were either covered or corollaries from Thm 1.8 in Folland, hence why I included it here - oh well (but I did give a slightly more concise proof for (b) :))

Proof.

(a) Since $(\cap_1^\infty B_i) \sqcup (B_1 \setminus \cap_1^\infty B_i) = B_1 \Rightarrow \mu(B_1) = \mu(\cap_1^\infty B_i) + \mu(B_1 \setminus \cap_1^\infty B_i) \Rightarrow \mu(\cap_1^\infty B_i) \leq \mu(B_1)$.

(b) Since $(B_1) \sqcup (\cup_2^\infty B_i \setminus B_1) = \cup_1^\infty B_i \Rightarrow \mu(\cup_1^\infty B_i) = \mu(B_1) + \mu(\cup_2^\infty B_i \setminus B_1) \Rightarrow \mu(\cup_1^\infty B_i) \geq \mu(B_1)$.

□

We now have all the necessary tools to prove the proposition as follows:

$$\mu\big(\liminf E_j\big) = \mu\left(\bigcup_{k=1}^{\infty}\left(\bigcap_{j=k}^{\infty} E_j\right)\right) \overset{*}{=} \lim_{k\to\infty} \mu\left(\bigcap_{j=k}^{\infty} E_j\right) = \liminf_{k\to\infty} \mu\left(\bigcap_{j=k}^{\infty} E_j\right) \overset{*}{\leq} \liminf_{k\to\infty} \mu\left(E_j\right)$$

Where $\overset{*}{=}$ is by μ's "Continuity from below" since $\cap_{j=k}^{\infty} E_j \subset \cap_{j=k+1}^{\infty} E_j$ $\forall k \in \mathbb{N}$, and $\overset{*}{\leq}$ is by Lem 2.1 (a).

$$\mu\big(\limsup E_j\big) = \mu\left(\bigcap_{k=1}^{\infty}\left(\bigcup_{j=k}^{\infty} E_j\right)\right) \overset{\star}{=} \lim_{k\to\infty} \mu\left(\bigcup_{j=k}^{\infty} E_j\right) = \liminf_{k\to\infty} \mu\left(\bigcup_{j=k}^{\infty} E_j\right) \overset{\star}{\geq} \liminf_{k\to\infty} \mu\left(E_j\right)$$

Where $\overset{\star}{=}$ is by μ's "Continuity from above" since $\cup_{j=k+1}^{\infty} E_j \subset \cup_{j=k}^{\infty} E_j$ $\forall k \in \mathbb{N}$, and $\overset{\star}{\leq}$ is by Lem 2.1 (b). □

1.7 Folland 1.9

Prove the following Proposition:

Proposition. 1.6:

If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Proof. Firstly, let us make the following observations:

$$(E\backslash F) \sqcup F = (E \cup F), \quad \text{and} \quad (E \cap F) \sqcup (E\backslash F) = E$$

Therefore, since μ is countably additive and therefore finitely additive, we can now see that:

$$\begin{aligned}
\mu(E) + \mu(F) &= \mu\big((E \cap F) \sqcup (E\backslash F)\big) + \mu(F) \\
&= \mu(E \cap F) + \mu(E\backslash F) + \mu(F) \\
&= \mu(E \cap F) + \mu\big((E\backslash F) \sqcup F\big) \\
&= \mu(E \cap F) + \mu(E \cup F)
\end{aligned}$$

□

1.8 Folland 1.10

Prove the following Proposition:

Proposition. 1.7:

Given a measure space, $(X, \mathcal{M}, \mu)$ and $E \in \mathcal{M}$, define $\mu_E(A) = \mu(A \cap E)$ for $A \in \mathcal{M}$. Then μ_E is a measure.

Proof. We first confirm that $\mu_E(\varnothing) = 0$ since $\mu_E(\varnothing) = \mu(\varnothing \cap E) = \mu(\varnothing) = 0$. Next, let $\{F_i\}_1^{\infty} \subset \mathcal{M}$ and $\{F_i\}_1^{\infty}$ disjoint. Then:

$$\mu_E\left(\bigcup_{i=1}^{\infty} F_i\right) = \mu\left(E \cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right) = \mu\left(\left(\bigcup_{i=1}^{\infty} E \cap F_i\right)\right) \overset{*}{=} \sum_{i=1}^{\infty} \mu\left(E \cap F_i\right) = \sum_{i=1}^{\infty} \mu_E\left(F_i\right)$$

Where $\overset{*}{=}$ since if $\{F_i\}_1^{\infty}$ is a disjoint family of sets, then $\{F_i \cap E\}_1^{\infty}$ will be as well. Thus, we have shown μ_E is indeed a measure. □

1.9 Folland 1.13

Prove the following Proposition:

Proposition. 1.8:

Every σ-finite measure is semi-finite.

Proof. Let μ be a σ-finite measure on the measurable space $(X, \mathcal{M})$. Firstly, if $\mu(X) < \infty$, μ will trivially be semi-finite. Therefore, suppose μ is σ-finite, but not finite. Now, let us arbitrarily pick $E \in \mathcal{M}$ s.t. $\mu(E) = \infty$ (we know at least one such element exists, namely X, since otherwise μ would be finite). From the definition of μ being σ-finite, we know that $\exists$ $\{F_i\}_1^\infty \subset \mathcal{M}$ s.t. $X = \cup_1^\infty F_i$ and $\mu(F_i) < \infty$ $\forall i \in \mathbb{N}$. One can easily see the following:

$$\mu(E) = \mu(E \cap X) = \mu\left(\bigcup_{i=1}^{\infty}(E \cap F_i)\right) \leq \sum_{i=1}^{\infty} \mu(E \cap F_i)$$

And since $\mu(E) = \infty$

$$\Rightarrow \infty \leq \sum_{i=1}^{\infty} \mu(E \cap F_i) \Rightarrow \sum_{i=1}^{\infty} \mu(E \cap F_i) = \infty$$

Furthermore, since $E \neq \varnothing$ (since otherwise $\mu(E) = 0 < \infty$) and $\mu(E) = \mu\left(\bigcup_{i=1}^{\infty}(E \cap X_i)\right)$, we know there must exist at least one $k \in \mathbb{N}$ s.t. $\mu(E \cap F_k) > 0$. On the other-hand, since $\mu(F_k) < \infty$ by construction, so too will $\mu(E \cap F_k) < \infty$. Therefore, since trivially $E \cap F_k \subset E$, we have shown that for an arbitrary $E \in \mathcal{M}$ s.t. $\mu(E) = \infty$, $\exists k \in \mathbb{N}$ s.t. $F_k \cap E \subset E$ and $\mu(F_k \cap E) < \infty$; I.e., all σ-finite measures are semi-finite.

$\square$

1.10 Folland 1.17

Prove the following Proposition:

Proposition. 1.9:

If μ^* is an outer measure on X and $\{A_j\}_1^\infty$ is a sequence of disjoint μ^*-measurable sets, then $\mu^*(E \cap (\cup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$ for any $E \subset X$.

Proof. Firstly, since μ^* is an outer measure, we know that:

$$\mu^*(E \cap (\cup_1^\infty A_j)) = \mu^*((\cup_1^\infty E \cap A_j)) \leq \sum_{j=1}^{\infty} \mu^*(E \cap A_j)$$

Now, let us define $B_n := \cup_1^n E_j$. Now, since A_j is μ^*-measurable $\forall j \in \mathbb{N}$, we know that $\forall n > 1$:

$$\mu^*(E \cap B_n) = \mu^*((E \cap B_n) \cap A_n) + \mu^*((E \cap B_n) \cap A_n^c) = \mu^*(E \cap A_n) + \mu^*(E \cap B_{n-1})$$

Therefore, iteratively using the above formula (by induction) for $B_n, \ldots, B_2$, and countable additivity being trivial for $n = 1$, we have shown that:

$$\mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu^*\left(E \cap A_j\right), \quad \forall n \in \mathbb{N}$$

Now, by monotonicty, we can easily see that:

$$\mu^*\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) \geq \mu^*\left(E \cap \bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu^*\left(E \cap A_j\right), \quad \forall n \in \mathbb{N}$$

And hence $\mu^*\left(E \cap \bigcup_{j=1}^{\infty} A_j\right) \geq \sum_{j=1}^{\infty} \mu^*\left(E \cap A_j\right)$. And thus since we shown both $\geq$ *and* $\leq$, we can conclude that $\mu^*(E \cap (\cup_1^\infty A_j)) = \sum_1^\infty \mu^*(E \cap A_j)$. □

1.11 Folland 1.18

Prove the following Proposition:

Proposition. 1.10:

Let $\mathcal{A} \subset \mathcal{P}(X)$ be an algebra, $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersections of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$ and μ^* the induced outer measure.

a) For any $E \subset X$ and $\epsilon > 0$, there exists $B \in \mathcal{A}_\sigma$ with $E \subset B$ and $\mu^*(B) \leq \mu^*(E) + \epsilon$

b) If $\mu^*(E) < \infty$, then E is μ^*-measurable $\iff$ there exists $C \in \mathcal{A}_{\sigma\delta}$ with $E \subset C$ and $\mu^*(C \backslash E) = 0$.

c) If μ_0 is σ-finite, the restriction $\mu^*(E) < \infty$ in (b) is superfluous.

Proof.

a) Let us recall the definition of $\mu^*(E)$ as:

$$\mu^*(E) := \inf\left\{\sum_{i=1}^{\infty} \mu_0(A_i) \;\middle|\; \{A_i\}_1^\infty \subset \mathcal{A}, E \subset \bigcup_{i=1}^{\infty} A_i\right\}$$

Therefore, by the definition of inf, $\forall \epsilon > 0\ \exists \{B_i\}_1^\infty$ s.t. $E \subset \cup_1^\infty B_i$ and $\sum_1^\infty \mu_0(B_i) \leq \mu^*(E) + \epsilon$. Therefore, if we define $B := \{B_i\}_1^\infty$ (same seq. as before), we note that $B \in \mathcal{A}_\sigma$, and also that:

$$\mu^*(B) \overset{*}{\leq} \sum_{i=1}^{\infty} \mu_0(A_i) \leq \mu^*(E) + \epsilon$$

Where $\overset{*}{\leq}$ because $\mu_0(B_i) = \mu^*(B_i)$, and B is μ^*-measruable.

b) Let us begin with the forward direction ($\mu^*(E) < \infty$, and E is μ^*-measurable). From part (a), we know $\exists\ \{C_i\} \subset \mathcal{A}_\sigma$ s.t. $E \subset C_k$ and $\mu^*(C_k) \leq \mu^*(E) + \frac{1}{k}\ \forall k \in \mathbb{N}$. Let us now define $C := \cap_1^\infty C_i$, to which we notice that $C \in \mathcal{A}_{\sigma\delta}$ and $E \subset C$ since $E \subset C_k\ \forall k \in \mathbb{N}$, and hence $\mu^*(E) \leq \mu^*(C)$. Furthermore, we note that since C_k is μ^*-measurable, so too will C_k^c, and hence $\cup_1^\infty C_i^c = (\cap_1^\infty C_i)^c = C^c$ is μ^*-measurable, and hence C is μ^*-measurable. Now, the following observation becomes apparent:

$$\mu^*(C) = \mu^*\left(\bigcap_{i=1}^{\infty} C_i\right) = \lim_{n\to\infty} \mu^*\left(\bigcap_{i=1}^{n} C_i\right) \leq \lim_{n\to\infty} \mu^*(C_n) = \mu^*(E)$$

Moreover, using the fact that $E \subset C$ the above now actually implies that $\mu^*(E) = \mu^*(C)$. We also recall that since E^c is μ^*-measurable, and since we already showed that C was μ^*-measurable, we can now also say that $C \cap E^c = B \backslash E$ is μ^*-measurable, and also note that hence:

$$\mu^*(C \backslash E) = \mu^*(C) - \mu^*(C \cap E) = \mu^*(C) - \mu^*(E) = 0$$

For the backward direction (there exists $C \in \mathcal{A}_{\sigma\delta}$ with $E \subset C$ and $\mu^*(C \backslash E) = 0$), first note that since $E \subset C$), $C = (B \backslash E) \cup E$. Next, since μ^* is the Carathèodory extension, $C \backslash E$ is μ^*-measurable. Therefore, we can easily conclude that $E = B \backslash (B \backslash E)$ is also μ^*-measurable.

c) Firstly, since μ_0 is σ-finite, we know that $\exists$ a disjoint set $\{X_i\}_1^\infty \subset \mathcal{A}$ s.t. $X = \sqcup X_i$ and $\mu_0(X_k) < \infty$ $\forall k \in \mathbb{N}$. Next, since $E \subset X$ is measurable, so too will $E_k := E \cap X_k$ $\forall k \in \mathbb{N}$, and by above and since $\{E \cap E_i\}_1^\infty$ is disjoint, we know that $E = \sqcup_1^\infty (E \cap X_i) = \sqcup_1^\infty E_i$, and naturally $\mu_0(E \cap X_k) < \infty$ $\forall k \in \mathbb{N}$. Since we are able to write E in this construction, E is μ^*-measurable. We can now figure out the following line of reasoning:

$$\begin{aligned} E\mu^*\text{-measurable} &\iff E_i\mu^*\text{-measurable} \\ &\iff \exists C_i \in \mathcal{A} \text{ s.t. } E_i \subset C_i, \mu^*(C_i \backslash E_i) = 0 \\ &\iff E \subset C = \bigcup_{i=1}^\infty C_i = \bigcup_{i=1}^\infty \left(\bigcap_{j=1}^\infty \left(\bigcup_{k=1}^\infty A_{ijk} \right) \right) \subset \bigcap_{i=1}^\infty \left(\bigcap_{j=1}^\infty \left(\bigcup_{k=1}^\infty A_{ijk} \right) \right) \in \mathcal{A}_{\sigma\delta} \end{aligned}$$

Where $\mu^*(C \backslash E) = \mu^*(\cup_1^\infty C_i \backslash E_i) \leq \sum_1^\infty \mu^*(C_i \backslash E_i) = \sum_1^\infty 0 = 0$. And hence $\mu * (E) < \infty$ did not matter if μ_0 is σ-finite.

□

1.12 Folland 1.26

Prove the following Proposition (by using Folland, Theorem 1.19):

Proposition. 1.11:

If $E \in \mathcal{M}_\mu$ and $\mu(E) < \infty$, then $\forall \epsilon > 0$ $\exists$ a set A that is a finite union of open intervals such that $\mu(E \triangle A) < \infty$.

Proof. We recall that by Theorem 1.18, $\exists \mathcal{U}^{open}$ s.t. $E \subset \mathcal{U}$ and $\mu(\mathcal{U}) \leq \mu(E) + \frac{1}{2}\epsilon$. Furthermore, by the inequality just stated, we know that $\mu(\mathcal{U}), \mu(E) < \infty$, and hence:

$$\mu(\mathcal{U} \backslash E) = \mu(\mathcal{U}) - \mu(E) < \frac{1}{2}\epsilon$$

Now, by recalling that all open sets in $\mathbb{R}$ can be written as $\sqcup_1^\infty \mathcal{U}_i$, we know that $\exists$ $\{\mathcal{U}_i\}_1^\infty$ s.t. $\sqcup_1^\infty \mathcal{U}_i = \mathcal{U}$. We now prove that actually:

$$\exists N \in \mathbb{N} \text{ s.t. } \mu(\mathcal{U}) = \mu\left(\sqcup_1^\infty \mathcal{U}_i = \mathcal{U}\right) < \mu\left(\sqcup_1^N \mathcal{U}_i\right) + \frac{1}{2}\epsilon$$

To see this, since $\{\mathcal{U}_i\}_1^\infty$ is disjoint:

$$\sum_{i=1}^\infty \mu\left(\mathcal{U}_i\right) = \mu(\mathcal{U}) < \mu(E) < \infty$$

Therefore, the series $\sum_1^\infty \mu\left(\mathfrak{U}_i\right)$ must converge, and hence, by the definition of convergent series', $\exists N \in \mathbb{N}$ s.t. $\sum_{N+1}^\infty \mu\left(\mu(\mathfrak{U}_i)\right) < \frac{1}{2}\epsilon$, and thus the inequality we sought to prove has now been shown.

Carrying on, let us define $\tilde{\mathfrak{U}} := \{\mathfrak{U}_i\}_1^N$. Since $\tilde{\mathfrak{U}} \subset \mathfrak{U} \Rightarrow \mu(\tilde{\mathfrak{U}}) \leq \mu(\mathfrak{U}) < \infty$ and also $\Rightarrow \tilde{\mathfrak{U}} \backslash E \subset \mathfrak{U} \backslash E$, hence:

$$\mu(\tilde{\mathfrak{U}} \backslash E) \leq \mu(\mathfrak{U} \backslash E) < \frac{1}{2}\epsilon$$

Now, also since $\mu(\tilde{\mathfrak{U}}) < \infty$, and since $\tilde{\mathfrak{U}} \subset \mathfrak{U} \Rightarrow E \backslash \tilde{\mathfrak{U}} \subset U \backslash \tilde{U}$, we can see that:

$$\mu(E \backslash \tilde{\mathfrak{U}}) \leq \mu(\mathfrak{U} \backslash \tilde{\mathfrak{U}}) = \mu(\mathfrak{U}) - \mu(\tilde{\mathfrak{U}}) = \sum_{i=N+1}^{\infty} \mu(\mathfrak{U}_i) < \frac{1}{2}\epsilon$$

Therefore, by combining the last two main inequalities, we have found a set $A = \tilde{\mathfrak{U}}$ which is a finite union of open intervals such that:

$$\mu(E \triangle \tilde{\mathfrak{U}}) = \mu(E \backslash \tilde{\mathfrak{U}}) + \mu(\tilde{\mathfrak{U}} \backslash E) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon$$

□

1.13 Folland 1.28

Prove the following Proposition:

Proposition. 1.12:

Let F be increasing and right continuous, and let μ_F be the associated measure. Then:

a) $\mu_F(\{a\}) = F(a) - F(a-)$

b) $\mu_F([a,b)) = F(b-) - F(a-)$

c) $\mu_F([a,b]) = F(b) - F(a-)$

d) $\mu_F((a,b)) = F(b-) - F(a)$

Proof.

a) We first note that we may construct $\{a\}$ from a countable intersection of h-intervals as follows:

$$\{a\} = \bigcap_{n=1}^{\infty} (a - 1/n, a]$$

Furthermore, since $(a - 1/n, a] \supset (a - 1/(n+1), a]$ $\forall n \in \mathbb{N}$, we may invoke continuity from above in that:

$$\mu_F(\{a\}) = \lim_{n\to\infty} \mu_F((a - 1/n, a]) = \lim_{n\to\infty} (F(a) - F(a - 1/n)) \overset{*}{=} F(a) - F(a-)$$

Where $\overset{*}{=}$ can be rigorously shown by noting that since F is an increasing function:

$$\lim_{n\to\infty} F(a - 1/n) = \sup\{F(x) \mid x < a\} = F(a-)$$

b) We first note that we may construct $[a,b)$ from a union of countable intersections and unions of h-intervals as follows:

$$[a,b) = [a,(a+b)/2] \cup (a,b) = \left(\bigcap_{n=1}^{\infty}(a-1/n,(a+b)/2]\right) \cup \left(\bigcup_{m=1}^{\infty}(a,b-1/m]\right)$$

Like in part a):

$$\begin{aligned}
\mu_F([a,(a+b)/2]) &= \lim_{n\to\infty} \mu_F\left(\bigcap_{n=1}^{\infty}(a-1/n,(a+b)/2]\right) \\
&= \lim_{n\to\infty} (F((a+b)/2) - F(a-1/n)) \\
&\overset{*}{=} F((a+b)/2) - F(a-)
\end{aligned}$$

Where $\overset{*}{=}$ is reasoned exactly as in a). Furthermore, since $(a,b-1/m) \subset (a,b-1/(m+1))$ $\forall m \in \mathbb{N}$, we may invoke continuity from below in that:

$$\begin{aligned}
\mu_F((a,b)) &= \lim_{n\to\infty} \mu_F\left(\bigcup_{n=1}^{\infty}(a,b-1/m]\right) \\
&= \lim_{n\to\infty} (F(b-1/m) - F(a)) \\
&\overset{\star}{=} F(b-) - F(a)
\end{aligned}$$

Where $\overset{\star}{=}$ can be rigorously shown by noting that since F is an increasing function:

$$\lim_{n\to\infty} F(b-1/m) = \sup\{F(x) \mid x < b\} = F(b-)$$

Therefore, since all the sets we've been dealing with so far have been bounded, we can see now that:

$$\begin{aligned}
\mu_F\big([a,b)\big) &= \mu_F\big([a,(a+b)/2]\big) + \mu_F\big((a,b)\big) - \mu_F\big((a,b) \cap [a,(a+b)/2]\big) \\
&= \mu_F\big([a,(a+b)/2]\big) + \mu_F\big((a,b)\big) - \mu_F\big((a,(a+b)/2]\big) \\
&= \Big[F\big((a+b)/2\big) - F\big(a-\big)\Big] + \Big[F\big(b-\big) - F\big(a\big)\Big] - \Big[F\big((a+b)/2\big) - F\big(a\big)\Big] \\
&= \Big[F(b-) - F(a-)\Big] + \Big[F\big((a+b)/2\big) - F\big((a+b)/2\big)\Big] + \Big[F\big(a\big) - F\big(a\big)\Big] \\
&= F(b-) - F(a-)
\end{aligned}$$

c) We first note that we may construct $[a,b]$ from countable intersection of h-intervals as follows:

$$[a,b] = \bigcap_{n=1}^{\infty}(a-1/n,b]$$

Thus, by making the change of variables of $(a+b)/2 \to b$, from the first half of b), we have already shown that $\mu_F([a,b]) = F(b) - F(a-)$.

d) We first note that we may construct (a,b) from a countable union of h-intervals as follows:

$$\bigcup_{n=1}^{\infty}(a,b-1/n]$$

Thus, from the second half of b), we have already shown that $\mu_F((a,b)) = F(b-) - F(a)$.

□

1.14 Folland 1.30

Prove the following Proposition:

Proposition. 1.13:

If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ $\exists$ an open interval $\hat{I}$ such that $m(E \cap I) > \alpha m(I)$.

Proof. If $\alpha \leq 0$, since $m(E) > 0 \Rightarrow \exists F \subset E$ s.t. $m(F) > 0$, and $F = (a, b], a < b$. If we thus take:

$$\hat{I} = \left(\frac{1}{4}(a+b),\ \frac{3}{4}(a+b)\right)$$

We have $m(E \cap I) = m(I) > 0 \geq \alpha m(I)$.

Now suppose $0 < \alpha < 1$. Since m is semi-finite, if $m(\hat{E}) = \infty$, we can simply take $E \subset \hat{E}$ s.t. $0 < m(E) < \infty$, and hence we actually restrict our problem to that of all E's s.t. $E \in \mathcal{L}$ and $0 < m(E) < \infty$. Let us also quickly note/recall that:

$$m(\{b\}) = 0 \Rightarrow m((a, b]) = m\big((a, b) \sqcup \{b\}\big) = m\big((a, b)\big) + m\big(\{b\}\big) = m\big((a, b)\big)$$

Now, for the sake of contradiction, assume $\forall I = (a, b), a < b$, we have: $m(E \cap I) \leq \alpha m(I)$. Let us choose $\epsilon_1 > 0$ so that $\epsilon_1 < \frac{1-\alpha}{\alpha}$ (and hence $\alpha(1 + \epsilon_1) < 1$). Moreover, from (Folland) Theorem 1.18, we know that $\forall \epsilon_2 > 0$ $\exists I = \sqcup_1^\infty (a_i, b_i)$ s.t. $E \subset I$ and $m(I) = \sum_1^\infty m\big((a_i, b_i)\big) < m(E) + \epsilon_2$. Next, from our discussion on (a, b) v.s. $(a, b]$, we can actually write $I = \sqcup_1^\infty (a_i, b_i]$, where I still satisfies everything that it did beforehand. Now, if we let $\epsilon_2 = m(E)\epsilon_1$, (which we can certainly do since $m(E) < \infty$), we see that:

$$m(I) = \sum_{i=1}^{\infty} (a_i, b_i] < m(E) + m(E)\epsilon_1 = m(E)(1 + \epsilon_1) < m(E)\left(1 + \frac{1-\alpha}{\alpha}\right) = m(E)\frac{1}{\alpha}$$
$$\Rightarrow \alpha m(I) < m(E)$$

Therefore, by combining the above inequality with our assumption in that $m(E \cap I_k) \leq \alpha m(I_k)$ $\forall k \in \mathbb{N}$, and that $E \subset I$, we see that:

$$m(E) = m(E \cap I) = \sum_{i=1}^{\infty} m(E \cap I_i) \leq \sum_{i=1}^{\infty} \alpha m(I_i) = \alpha m(I) < m(E)$$

Which is obviously a contradiction on the requirement of $m(E) > 0$, hence the converse must be true: I.e. our Proposition is true. $\square$

1.15 Folland 1.31

Prove the following Proposition:

Proposition. 1.14:

If $E \in \mathcal{L}$, and $m(E) > 0$, the set $\{E - E\} := \{x - y \mid x, y \in E\}$ contains an interval centered at 0. (If I is as in (Folland) Exercise 1.30, with $\alpha > \frac{3}{4}$, then $\{E - E\}$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$.)

Proof. From (Folland) 1.30, we know that $\exists$ I s.t. $\frac{3}{4}m(I) < m(E \cap I)$. Let us now define $F := E \cap I \subset E$, and naturally we will have $\{F - F\} \subset \{E - E\}$, hence if $\exists$ an interval centered at 0 in $\{F - F\}$, so too will that interval be in $\{E - E\}$.

We now claim that $F \cap \{F + x_0\} \neq \varnothing \Rightarrow x_0 \in \{F - F\}$. To see this, let $y \in F \cap \{F + x_0\} \Rightarrow y \in F$ and $\exists x \in F$ s.t. $y = x + x_0 \Rightarrow x_0 = y - x, y, x \in F \Rightarrow x_0 \in \{F - F\}$.

Trivially $0 \in \{F - F\}$ since $F \neq \varnothing$. Let us now let $z_0 \in \mathbb{R}$ s.t. $|z_0| < \frac{1}{2}m(I) < \frac{3}{4}m(I) < m(F)$. If we can show that $F \cap \{F + z_0\} \neq \varnothing \Rightarrow \big(-\frac{1}{2}m(I), \frac{1}{2}m(I)\big) \subset \{E - E\}$. Therefore, the remainder of this proof will be dedicated to showing $F \cap \{F + z_0\} \neq \varnothing$ where $|z_0| < \frac{1}{2}m(I)$.

Firstly, we note that:

$$m(I \backslash F) = m(I) - m(F) = m(I) - m(E \cap I) \leq m(I) - \frac{3}{4}m(F) = \frac{1}{4}m(F)$$

Furthermore, by applying the useful fact that $A \cap B = ((A \backslash C) \cap B) \cup ((C \backslash A) \cap B)$ twice, we find:

$$I \cap \{I + z_0\} = \big[F \cap \{F + z_0\}\big] \cup \big[F \cap (\{I \backslash F\} + z_0)\big] \cup \big[(I \backslash F) \cap \{I + z_0\}\big]$$

Our strategy now will be to show that $m\big(F \cap \{F + z_0\}\big) > 0$, which therefore would imply $I \cap \{I + z_0\}$ also has positive measure, and hence cannot be empty. To see this first note the following four properties:

$$\begin{aligned}
&m(I \cap \{I + z_0\}) \leq m\big[F \cap \{F + z_0\}\big] + m\big[F \cap (\{I \backslash F\} + z_0)\big] + m\big[(I \backslash F) \cap \{I + z_0\}\big] \\
&\text{and: } m\big[F \cap (\{I \backslash F\} + z_0)\big] \leq m\big[\{(I \backslash F) + z_0\}\big] = m\big[I \backslash F\big] \leq \frac{1}{4}m(F) \quad \text{from previously} \\
&\text{and: } m\big[F \cap (\{I \backslash F\} + z_0)\big] \leq m(I \backslash F) \leq \frac{1}{4}m(I) \quad \text{again} \\
&\text{and: } \frac{1}{2}m(I) < m(I) - |z_0| = m\big[I \cap \{I + z_0\}\big]
\end{aligned}$$

And hence combing all these we see that:

$$\frac{1}{2}m(I) < m\big[I \cap \{I + z)\}\big] \leq m\big[F \cap \{F + z_0\}\big] + \frac{1}{2}m(I) \quad \Rightarrow \quad m\big[F \cap \{F + z_0\}\big] > 0$$

□

1.16 Folland 1.33

Prove the following Proposition:

Proposition. 1.15:

There exists a Borel set $A \subset [0, 1]$ such that $0 < m(A \cap I) < m(I)$ for every sub-interval I of $[0, 1]$. (Hint: Every sub-interval of $[0, 1]$ contains Cantor-type sets of positive measure.)

Proof. The first observation we need to make is that since $|\mathbb{Q}| = \aleph_0 \Rightarrow |\mathbb{Q} \times \mathbb{Q}| = \aleph_0$ ($\aleph_0 :=$ "countably infinite"). Therefore, we can actually write the set of all closed sets I_k inside $[0, 1]$ where I_k's endpoints are rational numbers as a countable list: $\hat{I} = \{I_j\}_1^\infty$. By the hint, we know that every sub-interval of $[0, 1]$ contains Cantor-type sets (which will certainly have rational endpoints). Our plan will therefore be through induction, to explicitly describe a Borel set made up of necessary Cantor-like sets which will satisfy the needed inequality.

Let A_k, B_k be strict subsets of I_k (which we can do because we're assuming $I \neq \varnothing$, and due to the density of the rationals) s.t. $A_i \cap B_k = \varnothing$ and $m(A_i), m(B_j) > 0$ $\forall i, j \leq N$. We can therefore define:

$$C_N := I_N \backslash \bigsqcup_{j=1}^{N} (A_j \cup B_j)$$

And therefore, we can find a Cantor-type set D_N and $\tilde{D}_N$ s.t. $m(D_N), D(\tilde{D}_N) > 0\ \forall N \in \mathbb{N}$. If we let $D := \cup_{N=1}^{\infty} D_N$, then $\forall$ sub-intervals $I \subset [0,1]$, $\exists N$ s.t. $I_N \subset I$ and we will have:

$$0 < m(D_N) \leq m(D \cap I_N) \leq m(D \cap I) \leq m(D \cap I) + m(\tilde{D}_N) \leq m(I)$$

I.e., by seeing that $A = D$ $0 < m(I \cap D) < m(I)$. □

2 Chapter 2

2.1 Folland 2.1

Prove the following Proposition:

Proposition. 2.1:

Let $f : X \to \overline{\mathbb{R}}$ and $Y = f^{-1}(\mathbb{R})$. Then f is measurable $\iff$ $f^{-1}(\{-\infty\}) \in \mathcal{M}$, $f^{-1}(\{\infty\}) \in \mathcal{M}$, and f is measurable on Y.

Proof. To be clear on notation, if $X = \{\pm\infty\}$, then either $X = \{\infty\}$ or $X = \{-\infty\}$, and naturally $\{-\infty, \infty\} \neq X$.

For the forward direction, since f is measurable and $\{\pm\infty\} \in \mathcal{B}_{\overline{\mathbb{R}}}$, it implies $f^{-1}(\{\pm\infty\}) \in \mathcal{M}$. Furthermore, again by f's is measurability and since $\mathbb{R} \in \mathcal{B}_{\overline{\mathbb{R}}}$, it implies $f^{-1}(\mathbb{R}) \in \mathcal{M}$. Therefore, we may conclude that if $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, then $f^{-1}(B) \in \mathcal{M}$ and $f^{-1}(B) \cap f^{-1}(\mathbb{R}) = f^{-1}(B) \cap Y \in \mathcal{M}$, I.e., f is measurable on Y.

For the converse, if we let $B \in \mathcal{B}_{\overline{\mathbb{R}}}$, then we can see that:

$$f^{-1}(B) = \left(f^{-1}(B) \cap f^{-1}(\mathbb{R})\right) \sqcup \left(f^{-1}(B) \cap f^{-1}(\overline{\mathbb{R}} \backslash \mathbb{R})\right)$$

And since $f^{-1}(\mathbb{R})$ is measurable, naturally $f^{-1}(B) \cap f^{-1}(\mathbb{R}) = f^{-1}(B \cap \mathbb{R})$ is as well. Next, we note that:

$$f^{-1}(B) \cap f^{-1}(\overline{\mathbb{R}} \backslash \mathbb{R}) = f^{-1}(B) \cap f^{-1}(\{-\infty, \infty\}) = f^{-1}(B \cap \{-\infty, \infty\})$$

Which naturally is either $f^{-1}(\varnothing) = \varnothing, f^{-1}(\{-\infty\}), f^{-1}(\{\infty\})$, or $f^{-1}(\{-\infty, \infty\} = f^{-1}(\{-\infty\}) \cup f^{-1}(\{\infty\})$, all of which are measurable since $f^{-1}(\{-\infty\})$ and $f^{-1}(\{\infty\})$ are by assumption measurable. Combining these two implications of our assumptions, we can see f is measurable since:

$$f^{-1}(B) = \left(f^{-1}(B) \cap f^{-1}(\mathbb{R})\right) \sqcup \left(f^{-1}(B) \cap f^{-1}(\overline{\mathbb{R}} \backslash \mathbb{R})\right) \in \mathcal{M}$$

□

2.2 Folland 2.2

Prove the following Proposition:

Proposition. 2.2:

Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

a) fg is measurable (where $0 \cdot (\pm\infty) = 0$).

b) Fix $a \in \overline{\mathbb{R}}$, and define $h(x) = a$ if $f(x) = -g(x) = \pm\infty$, and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Proof. We actually do this problem in reverse ordering.

b) We prove this fact by separating the problem into 2 lemmas, and one final main result:

For the first mini-lemma, we note that $A_\infty := \{x \in X \mid f(x) = -g(x) = \pm\infty\}$ is measurable since f and g are measurable.

For the second mini lemma, we make the observation that:

$$h^{-1}(\{\infty\}) = (f+g)^{-1}(\{\infty\}) = \Big(f^{-1}(\{\infty\}) \cap g^{-1}((-\infty,\infty])\Big) \cup \Big(f^{-1}((-\infty,\infty]) \cap g^{-1}(\{\infty\})\Big)$$

Since $h(x) = \infty \iff$ either $[f(x) = \infty$ and $g(x) > -\infty]$ or $[g(x) = \infty$ and $f(x) > -\infty]$, or $[f(x) = g(x) = \infty]$. Similarly for the $\{-\infty\}$ (sub-) case:

$$h^{-1}(\{-\infty\}) = (f+g)^{-1}(\{-\infty\}) = \Big(f^{-1}(\{-\infty\}) \cap g^{-1}([-\infty,\infty))\Big) \cup \Big(f^{-1}([-\infty,\infty)) \cap g^{-1}(\{-\infty\})\Big)$$

Since $h(x) = -\infty \iff$ either $[f(x) = -\infty$ and $g(x) < \infty]$ or $[g(x) = -\infty$ and $f(x) < \infty]$, or $[f(x) = g(x) = -\infty]$. We naturally recognize the above to certainly be measurable (again) since f and g are measurable.

Now for the final main result. Let $b \in \mathbb{R}$, then:

$$h^{-1}((b,\infty]) = h^{-1}((b,\infty)) \cup h^{-1}(\{\infty\})$$

Since we already showed that $h^{-1}(\{\infty\})$ is measurable, we now seek to show that $h^{-1}((b,\infty))$ is measurable. This can be seen since:

$$\begin{aligned} h^{-1}((b,\infty)) &= \begin{cases} (f+g)^{-1}((b,\infty)) & \text{if } a \le b \\ (f+g)^{-1}((b,a)) \cup (h)^{-1}(\{a\}) \cup (f+g)^{-1}((a,\infty)) & \text{if } a > b \end{cases} \\ &= \begin{cases} (f+g)^{-1}((b,\infty)) & \text{if } a \le b \\ A_\infty \cup (f+g)^{-1}((b,\infty)) & \text{if } a > b \end{cases} \end{aligned}$$

Where we already showed that A_∞ is measurable, and by f and g's measurability, all the sets above which make up $h^{-1}((b\infty))$ are measurable, and hence $h^{-1}((b,\infty])$ is measurable; therefore, h is measurable.

a) Let us define $\mathbb{Q}^+ := \{r \in \mathbb{Q} \mid r > 0\}$ and $\mathbb{Q}^- := \{r \in \mathbb{Q} \mid r < 0\}$, which is a subsets of $\mathbb{Q}$ and hence countable. Suppose now that $f, g \ge 0$, if $a \ge 0$, then we will have:

$$\begin{aligned} (fg)^{-1}((a,\infty]) &= \{x \in X \mid f(x)g(x) > a\} \\ &= \bigcup_{r \in \mathbb{Q}^+} \Big(\{x \in X \mid f(x) > r\} \cap \{x \in X \mid g(x) > a/r\}\Big) \end{aligned}$$

Furthermore, if $a < 0$, (since $f, g \ge 0$) we have:

$$(fg)^{-1}((a,\infty]) = \{x \in X \mid f(x)g(x) > a\} = X$$

Therefore, since irregardless of a, $(fg)^{-1}((a,\infty])$ is a countable union of measurable sets, fg is measurable for $f, g \ge 0$. Our strategy henceforth will be to write $f = f^+ - f^-_*$ and $g = g^+ - g^-_*$, where $f^+ := \max(0,f)$, $f^-_* := -\min(0,f)$, and similarly for g. Therefore, we naturally have:

$$fg = (f^+ - f^-_*)(g^+ - g^-_*) = \big(f^+g^+ + f^-_* g^-_*\big) + \big(-\big(f^+g^-_* + f^-_* g^+\big)\big)$$

Now, by our previous work, since $f^+, g^+, f^-_*, g^-_* \ge 0$, it follows that the first half of the above expression is measurable (since by part b, we showed that the addition of two measurable functions as defined in this question is measurable). And also recalling that f measurable $\iff$ $-f$ measurable, we can therefore conclude that fg is indeed measurable.

$\square$

2.3 Folland 2.3

Prove the following Proposition:

Proposition. 2.3:

If $\{f_n\}$ is a sequnce of measurable functions on X, then $\{x \mid \lim f_n(x) \text{ exists}\}$ is a measurable set.

Proof. We first recall that by (Folland) Proposition 2.7, when $\{f_n\}$ is defined as in the question, $g_3(x) = \limsup_{n\to\infty} f_n(x)$ and $g_4(x) = \liminf_{n\to\infty} f_n(x)$ are both measurable. If, as in Exercise 2.2, we let $a = 1$, then function $g_3 - g_4$ is measurable (and is equal to 1 when $g_3 = g_4 = \pm\infty$). Finally, by noting that $\lim f_n(x)$ exists $\iff g_3 = g_4$, we can actually write:

$$\{x \in X \mid \lim f_n(x) \text{ exists}\} = Kernel(g_3 - g_4) = \{x \in X \mid g_3(x) = g_4(x)\} = (g_3 - g_4)^{-1}(0)$$

Which is most certainly measurable since g_3 and g_4 are measurable, and the difference of such measurable functions is also measurable (Corollary of Exercise 4.2 by combining the fact that f measurable $\iff$ $-f$ measurable, and taking $f - g = f + (-g)$). $\square$

2.4 Folland 2.4

Prove the following Proposition:

Proposition. 2.4:

If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Proof. Firstly, by the density of the rationals, $(a, \infty] = \cup_{r\in\mathbb{Q}_r^+}(r, \infty]$, where $a \in \mathbb{R}$, and $\mathbb{Q}_a^+ := \{r \in \mathbb{Q} \mid r > a\}$. Naturally since $\mathbb{Q}_a^+$ is countable and $\mathcal{B}_{\overline{\mathbb{R}}}$ is generated by the intervals in the form of $(a, \infty]$, and since:

$$f^{-1}((a, \infty]) \subset \bigcup_{r\in\mathbb{Q}_a^+} f^{-1}((r, \infty]) \in \mathcal{M}$$

By (Folland) Proposition 2.1, it follows that f is measurable. $\square$

2.5 Folland 2.7

Prove the following Proposition:

Proposition. 2.5:

Suppose that for each $\alpha \in \mathbb{R}$ we are given a set $E_\alpha \in \mathcal{M}$ such that $E_\alpha \subset E_\beta$ whenever $\alpha < \beta$, $\cup_{\alpha\in\mathbb{R}} E_\alpha = X$, and $\cap_{\alpha\in\mathbb{R}} E_\alpha = \varnothing$. Then there is a measurable function $f : X \to \mathbb{R}$ such that $f(x) \le \alpha$ on E_α and $f(x) \ge \alpha$ on E_α^c for every α. (Use (Folland) Exercises 2.4).

Proof. We claim that $f(x) := \inf\{\alpha \in \mathbb{R} \mid x \in E_\alpha\}$, where E_α has the same construction as given in the Proposition, will satisfy the requirements of being measurable and the stated inequalities. We begin first by showing the latter.

Suppose $x \in E_\alpha$, then by the construction of f, we immediately have $f(x) \le \alpha$. Now, suppose $\alpha \in E_\alpha^c$, then $\forall \beta \le \alpha$, $E_\alpha^c \subset E_\beta^c$ since $E_\beta \subset E_\alpha$; therefore, $x \in E_\beta^c \Rightarrow x \notin E_\beta\ \forall \beta \le \alpha \Rightarrow f(x) \ge \alpha$ if $x \in E_\alpha^c$.

Again by the construction of f, it is clear that $\cup_{\alpha\in\mathbb{R}} E_\alpha = X$ and $\cup_{\alpha\in\mathbb{R}} E_\alpha^c = X$. From this, given $\forall x \in X$, we know that $\exists\ \alpha, \beta \in \mathbb{R}$ such that $x \in E_\alpha$ and $x \in E_\beta$ and most importantly since $\alpha, \beta \in \mathbb{R}$:

$$-\infty < \alpha \leq f(x) \leq \beta < \infty$$

And hence $f(x) \neq \pm\infty$ irregardless of x. It'll now be a lot easier to conclude measurability since we no longer have to worry about the possibility that $f(x) = \pm\infty$.

Let us now take $r \in \mathbb{Q}$, and note that by first set of inequalities established, if $x \in X$, then $f(x) < r \iff \exists q \in \mathbb{R}$ s.t. $x \in E_q$. Equivalently: $f^{-1}((-\infty, r)) = \cup_{q<r} E_\alpha$. By the density of $\mathbb{Q}$, we can actually restrict that $q, r \in \mathbb{Q}$. We therefore have:

$$f^{-1}((-\infty, r)) = \bigcup_{q<r} E_q, \quad \text{where } q, r \in \mathbb{Q}$$

And since $E_q \in \mathcal{M}\ \forall q$, and since $\{q \in \mathbb{Q} \mid q < r\}$ is a countable set, we naturally have $f^{-1}((-\infty, r)) \in \mathcal{M}$. Furthermore, by the inequalities established, we also have:

$$f^{-1}([r, \infty)) = \bigcup_{q<r} E_r^c \in \mathcal{M}$$

And since we showed this to be true $\forall r \in \mathbb{Q}$, by Exercise 4.4, f is measurable.

□

2.6 Folland 2.8

Prove the following Proposition:

Proposition. 2.6:

If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Proof. We first state our strategy: If we can show that $\forall a \in \mathbb{R}$, $f^{-1}([am\infty))$ is an interval, then f must be Borel measurable., let us note that as trivial corollary of (Folland) Proposition 2.3, f measurable $\iff -f$ measurable. Thus, without loss of generality, assume f is monotone increasing. Suppose now that $a \in \mathbb{R}$, $x \in f^{-1}([a, \infty)$, and $y \in [x, \infty)$. Therefore, since f is monotone increasing:

$$a \leq f(x) \leq f(y) \Rightarrow y \in f^{-1}([a, \infty))$$

Since this is true $\forall x, y \in [a, \infty)$, it actually proves that $f^{-1}([a, \infty))$ is indeed an interval, and therefore Borel measurable, and hence f is Borel measurable since this is true $\forall a \in \mathbb{R}$. □

2.7 Folland 2.9

Prove the following Proposition:

Proposition. 2.7:

Let $f : [0,1] \to [0,1]$ be the Cantor Function (Folland Section 1.5), and let $g(x) = f(x) + x$.

a) g is a bijection from $[0,1]$ to $[0,2]$, and $h = g^{-1}$ is continuous from $[0,2]$ to $[0,1]$.

b) If C is the Cantor set, $m(g(C)) = 1$.

c) By (Folland) Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue non-measurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.

d) There exist a Lebesgue measurable function F and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.

Proof.

a) We first recall (from Folland) that the Cantor Function, $f(x)$ is monotone increasing, and naturally $h(x) = x$ is a strictly increasing function, and hence $g(x) = f(x) + x$ is also strictly increasing and therefore injective. Next, to show surjectivity, note that g is a continuous function, and $g(0) = f(0) + 0 = 0$, and $g(1) = f(1) + 1 = 2$; hence, by the intermediate value theorem, g is surjective.

   We now have all the necessary components to conclude that g is a bijection, and since g is a continuous bijective function, and $[0,1]$ is compact, g's inverse, g^{-1}, is continuous from $[0,2]$ to $[0,1]$.

b) Firstly, by g's surjectivity, and C being measurable, we see that:

$$g([0,1]\backslash C) \sqcup g(C) = g([0,1] \cap C^c) \sqcup g(C) = [0,2] \quad \Rightarrow \quad m(g(C)) + m(g([0,1]\backslash C)) = 2$$

   Next, since C is a closed set $\Rightarrow [0,1]\backslash C$ is an open set. Therefore, since all open subsets of $[0,1]$ may be written as a countable union of disjoint open sets, let us write $[0,1]\backslash C = \sqcup_1^\infty \mathcal{O}_j, \mathcal{O}_j = (a_j, b_j)$. Now, since f is by construction constant on $[0,1]\backslash C$, and recalling that $m(C) = 0 \Rightarrow m([0,1]\backslash C) = 1 \Rightarrow m(\sqcup_1^\infty \mathcal{O}_j) = 1$, we see:

$$\begin{aligned} m(g([0,1]\backslash C)) &= m\left(g\left(\bigsqcup_{j=1}^{\infty} \mathcal{O}_j\right)\right) = \sum_{j=1}^{\infty} m(g(\mathcal{O}_j)) \\ &= \sum_{j=1}^{\infty} \Big(m(f(b_j) - f(a_j)) + m(b_j - a_j) \Big) \\ &= \sum_{j=1}^{\infty} m(\mathcal{O}_j) && \text{since } f(b_j) = f(a_j) \ \forall j \in \mathbb{N} \\ &= m\left(\bigsqcup_{j=1}^{\infty} \mathcal{O}_j\right) \\ &= 1 \end{aligned}$$

   And hence $m(g(C)) = 1$ by the the first part of this proof.

c) To show Lebesgue measurability, naturally $B \subset C$, and since C is measurable with measure $m(C) = 0$, it implies $m(B) \leq m(C) = 0$, and hence Lebesgue measurable since null sets are measurable.

For the sake of contradiction, suppose $B = g^{-1}(A)$ is Borel measurable. In part a), we showed that g^{-1} is continuous and bijective; therefore $g(B) = g(g^{-1}(A)) = A$. However, by the continuity of g, if $g^{-1}(A)$ was Borel, so too would $g(g^{-1}(A)) = A$, hence a contradiction since A is not Lebesgue measurable; therefore, B cannot be Borel measurable.

d) Let $F = \chi_B$; I.e., $F(x) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{if } x \in B^c \end{cases}$, and also set $G = g^{-1}$. Naturally G is Lebesgue measurable since it is continuous, we now wish to prove that so too is F. This can be seen by noticing $F^{-1}((a,\infty)) = \varnothing$ or B or $\mathbb{R}$, but all these possibilities are Lebesgue measurable, hence F is Lebesgue measurable. We can now look at the following reasoning:

$$\begin{aligned}(F \circ G)^{-1}((1/2,\infty)) = G^{-1} \circ F^{-1}([1/2,\infty)) &= \{x \in [0,2] \mid \chi_B(g^{-1}(x)) \in [1/2,\infty)\} \\ &= \{x \in [0,2] \mid g^{-1}(x) \in B\} \\ &= G^{-1}(B) = g(g^{-1}(A)) = A\end{aligned}$$

Now since A is not Lebesgue measurable, $F \circ G$ also will not be Lebesgue measurable.

□

2.8 Folland 2.10

Prove the following Proposition:

Proposition. 2.8:

The following implications are valid $\iff$ the measure μ is complete:

a) If f is measurable and $f = g$ μ-a.e., then g is measurable.

b) If f_n is measurable for $n \in \mathbb{N}$ and $f_n \to f$ μ-a.e., then f is measurable.

Proof.

a) For the forward direction, suppose a) holds. Then let $N \in \mathcal{M}$ be a measurable set s.t. $\mu(N) = 0$, and $N_1 \subset N$. If we define $f :\equiv 0$ and $\chi_{N_1} := 1$ if $x \in N_1$, and 0 otherwise, then trivially f is measurable and $f = \chi_{N_1}$ μ-a.e., so by our assumptions g is measurable. Now, by noting that $\chi_{N_1}^{-1}(\{1\}) = N_1 \in \mathcal{M}$ by g's measurability, and since this is true $\forall N_1 \subset N$, we have arrived at the definition of μ being complete.

For the backward direction, suppose μ is complete, and let f be measurable and $f = g$ μ-a.e. Explicitly, let $N \in \mathcal{M}$ be the measurable set s.t. $\mu(N) = 0$ and $f(x) = g(x)$ $\forall x \in N^c$. Then if A is measurable, we have:

$$g^{-1}(A) = \left[g^{-1}(A) \cap N\right] \sqcup \left[g^{-1}(A) \cap N^c\right] = \left[g^{-1}(A) \cap N\right] \sqcup \left[f^{-1}(A) \backslash N\right]$$

Looking at the right hand side, we can see $g^{-1}(A) \cap N \subset N$ is measurable by the definition of μ being a complete measure since $\mu(N) = 0$. Furthermore, $f^{-1}(A) \backslash N \subset f^{-1}(A)$ since f is measurable. With these two facts, we may therefore conclude that g is indeed measurable.

b) For the forward direction, suppose b) holds. Then let $N \in \mathcal{M}$ be a measurable set s.t. $\mu(N) = 0$, and $N_1 \subset N$. If we let $f_n = 0$ and χ_{N_1} as before, then like in the forward direction of a), we have $f_n \to \chi_{N_1}$ μ-a.e., so χ_{N_1} is measurable. Therefore, $\chi_{N_1}^{-1}(\{1\}) \in \mathcal{M}$, and since this is true $\forall N_1 \subset N$, we have arrived at the definition of μ being complete.

For the backward direction, suppose μ is complete, and f_n is measurable $\forall n \in \mathbb{N}$, and $f_n \to f$ μ-a.e. By (Folland) Proposition 2.7, $g_3(x) = \limsup_{j\to\infty} f_j(x)$ is measurable since f_n is measurable $\forall n \in \mathbb{N}$. Furthermore, since $f_n \to f$ μ-a.e., we have $g_3 = f$ μ-a.e., and thus by the backward direction of part a) above, f is measurable.

□

2.9 Folland 2.12

Prove the following Proposition:

Proposition. 2.9:

If $f \in L^+$ and $\int f < \infty$, then $\{x \mid f(x) = \infty\}$ is a null set and $\{x \mid f(x) > 0\}$ is σ-finite.

Proof. Let $E := \{x \mid f(x) = \infty\}$, $F := \{x \mid f(x) > 0\}$, $F_n := \{x \mid f(x) > 1/n\}$, and f satisfy $f \in L^+$ and $\int f < \infty$. Let us now define the two sets of functions $\{\phi_n\}_1^\infty$ and $\{\varphi_n\}_1^\infty$, where $\phi_n = n\chi_E$ and $\varphi_n = \chi_{F_n}/n$.

To prove E is a null set, we make the observation that since $f(x) = \infty$ $\forall x \in E$, and $\chi_n(x) < \infty$ $\forall n \in \mathbb{N}$, we have:

$$\begin{aligned} 0 \leq \phi_n(x) \leq f(x)\ \forall x \in X \quad &\Rightarrow \quad n\mu(E) = \int \phi_n \; d\mu \leq \int f \; d\mu \\ &\Rightarrow \quad \mu(E) \leq \frac{1}{n} \int f \; d\mu \end{aligned}$$

Thus, since $\int f \; \delta\mu < \infty$, letting $n \to \infty$, we see that $\mu(E) = 0$; I.e., E is a null set.

By the construction of $\{F_n\}_1^\infty$, we have $\cup_1^\infty F_n$, so to conclude that F is σ-finite, we simply need to show that $\mu(F_n) < \infty$ $\forall n \in \mathbb{N}$. This is easily ascertained since $f(x) > 1/n$ $\forall x \in F_n$, and $\int f < \infty$, we have:

$$\begin{aligned} 0 \leq \varphi_n(x) \leq f(x)\ \forall x \in F_n \quad &\Rightarrow \quad \frac{1}{n}\mu(F_n) = \int \varphi_n \; d\mu \leq \int f \; d\mu \\ &\Rightarrow \quad \mu(F_n) \leq n \int f \; d\mu < \infty \end{aligned}$$

And hence $\mu(F_n) < \infty$ $\forall n \in \mathbb{N}$, which implies F is σ-finite.

□

2.10 Folland 2.13

Prove the following Proposition:

Proposition. 2.10:

Suppose $\{f_n\}_1^\infty \subset L^+$, $f_n \to f$ pointwise, and $\int f = \lim \int f_n < \infty$. Then $\int_E f = \lim \int_E f_n$ $\forall E \in \mathcal{M}$. However, this need not be true if $\int f = \lim f_n = \infty$.

Proof. Let $E \in \mathcal{M}$ and $\int f < \infty$, and so we define χ_E s.t. $\int_E f = \int \chi_E f$, and so we have:

$$\int_E f = \int \chi_E f \leq \int f = \lim \int f_n < \infty$$

Furthermore, by (Folland) Theorem 2.15, we have:

$$\int f = \int (\chi_E f + \chi_{E^c} f) = \int \chi_E f + \int \chi_{E^c} f$$

And similarly for substituting f_n for f above. Now, since $f_n \to f \Rightarrow \chi_F f_n \to \chi_F f$ $\forall F \in \mathcal{M}$, we may apply Fatou's Lemma as follows:

$$\int_E f = \int \liminf_{n\to\infty} \chi_E f_n \le \liminf_{n\to\infty} \int_E f_n \overset{*}{=} \liminf_{n\to\infty} \left(\int f_n - \int_{E^c} f_n \right) \overset{**}{=} \int f - \limsup_{n\to\infty} \int_{E^c} f_n$$

Where we have $\overset{*}{=}$ since $\int_E f = \int \chi_E f + \int \chi_{E^c} f$, and $\overset{**}{=}$ since $\liminf \int f_n = \lim \int f_n = \int f$ and $\liminf -\int g = -\limsup \int g$. However, since all terms above are finite, we may gain by apply Fatou's Lemma (and in noticing the similarity to the steps made above) to see that:

$$\limsup_{n\to\infty} \int_{E^c} f_n = \limsup_{n\to\infty} \left(\int f_n - \int_E f_n \right) = \int f - \liminf_{n\to\infty} \int_E f_n \le \int f - \int_E f$$

And thus by substituting this in, we have:

$$\int_E f \le \liminf_{n\to\infty} \int_E f_n \le \limsup_{n\to\infty} \int_E f_n \le \int_E f$$

And therefore all the inequalities in the equation(s) above are actually equalities, and so we have:

$$\liminf_{n\to\infty} \int_E f_n = \limsup_{n\to\infty} \int_E f_n = \lim_{n\to\infty} \int_E f = \int_E f$$

We now turn our attention showing the above result need not hold if $\int f = \lim \int f = \infty$ by means of a counter-example. Let $E = (0,1]$, $f = \chi_{[2,\infty)}$, and $f_n = \chi_{[2,\infty)} + n\chi_{(0,1/n]}$. Then $f_n \to f$ p.w., and:

$$\int_{(0,1]} f_n = n\mu((0,1/n]) = 1 \ \forall n \in \mathbb{N} \quad \Rightarrow \quad \lim_{n\to\infty} \int_{(0,1]} f_n = 1$$

However, $\int_{(0,1]} f = 0$, thus $\int_E f = \lim \int_E f_n$ need not be true if $\lim \int f = \int f = \infty$. $\square$

2.11 Folland 2.14

Prove the following Proposition:

Proposition. 2.11:

If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$. (First Suppose that g is simple.)

Proof. Trivially, since $f \in L^+$, we have that $\lambda(E) \ge 0$ $\forall E \in \mathcal{M}$. Moreover, one can see that $\lambda(\varnothing) = 0$:

$$\lambda(\varnothing) = \int_\varnothing f \, d\mu = \int \chi_\varnothing f \, d\mu = 0$$

To fully show that λ is a measure on $\mathcal{M}$, we need that for any disjoint sequence of sets, $\{E_j\}_1^\infty \in \mathcal{M}$, $\lambda(\sqcup_1^\infty E_j) = \sum_1^\infty \lambda(E_j)$. We can deduce this fact from the following:

$$\begin{aligned}
\lambda\Big(\bigsqcup_{j=1}^{\infty} E_j\Big) &= \int_{\sqcup_1^\infty E_j} f \, d\mu = \int \chi_{(\sqcup_1^\infty E_j)} f \, d\mu \\
&= \int \Big(\sum_{j=1}^{\infty} \chi_{E_j}\Big) f \, d\mu \overset{*}{=} \sum_{j=1}^{\infty} \int \chi_{E_j} f \, d\mu \qquad\qquad \overset{*}{=} \text{ by (Folland) Theorem 2.15} \\
&= \sum_{j=1}^{\infty} \int_{E_j} f \, d\mu = \sum_{j=1}^{\infty} \lambda(E_j)
\end{aligned}$$

We have thus shown all the necessary conditions for λ to be a measure do indeed hold.

Next, let $g \in L^+$, and assume that g is simple $\Rightarrow g = \sum_1^n a_j \chi_{E_j}$. Therefore:

$$\begin{aligned}
\int g \, d\lambda = \sum_{j=1}^{n} a_j \lambda(E_j) = \sum_{j=1}^{n} \int_{E_j} f \, d\mu = \sum_{j=1}^{n} \int \chi_{E_j} f \, d\mu \\
\overset{*}{=} \int \Big(\sum_{j=1}^{n} a_j \chi_{E_j}\Big) f \, d\mu = \int g f \, d\mu \qquad \overset{*}{=} \text{ by (Folland) Theorem 2.15}
\end{aligned}$$

And so we get the required result when g is simple. However, by (Folland) Theorem 2.10, we know that since $f \in L^+$, $\exists \{\phi_n\}_1^\infty$ s.t. $0 \le \phi_1 \le \phi_2 \le \cdots \le f$, $\phi_n \to f$ p.w., and $\phi_n \to f$ uniformly on any set on which f is bounded. Therefore, we can apply the Monotone Convergence Theorem (used if $\overset{\star}{=}$ denoted) as follows:

$$\int g \, d\lambda \overset{\star}{=} \lim_{n\to\infty} \int \phi_n \, d\lambda \overset{*}{=} \lim_{n\to\infty} \int \phi_n f \, d\mu \overset{\star}{=} \int g f \, d\mu \qquad \overset{*}{=} \text{ since } \phi_n \text{ simple } \forall n \in \mathbb{N}$$

□

2.12 Folland 2.16

Prove the following Proposition:

Proposition. 2.12:

If $f \in L^+$ and $\int f < \infty$, $\forall \epsilon > 0\ \exists E \in \mathcal{M}$ s.t. $\mu(E) < \infty$ and $\int_E f > \left(\int f\right) - \epsilon$

Proof. Firstly, By (Folland) Exercise 2.12 (proved above - 5.2), we know that $F := \{x \mid f(x) > 0\}$ is σ-finite. In the proof of (Folland) 2.12, we showed that $F_n := \{x \mid f(x) > 1/n\}$ has the nice properties of $\mu(F_n) < \infty$ and $\cup_1^\infty F_n = F$. Furthermore, it is also apparent from the construction of F_n that $F_n \subset F_{n+1}\ \forall n \in \mathbb{N}$ - I.e., $\{F_n\}_1^\infty$ is monotone increasing, and so $\{\chi_{F_n}\}_1^\infty$ will be an increasing sequence in L^+ s.t. $\chi_{F_n} \le \chi_{F_{n+1}}\ \forall n \in \mathbb{N}$, and $\lim_{n\to\infty} \chi_{F_n} = \chi_F$.

Since $\{\chi_{F_n}\}_1^\infty$ and χ_F satisfy necessary conditions for the Monotone Convergence Theorem, and in noticing $\int f = \int \chi_F f$, we may apply it as follows:

$$\int f = \int \chi_F f = \lim_{n\to\infty} \int \chi_{F_n} f = \int_{F_n} f$$

We also note that, since $\chi_{F_n} \subset \chi_F \ \forall n \in \mathbb{N}$, we have:

$$\int f = \int \chi_F f \leq \int \chi_{F_n} f = \int_{F_n} f \quad \forall n \in \mathbb{N}$$

Therefore, $\int_{F_n}$ is an increasing sequence with the limit of $\int f$. So by this convergence, we have $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that:

$$\int_{F_N} f > \left(\int f \right) - \epsilon$$

I.e., we have proven the existence of an $F_N = E \in \mathcal{M}$ which satisfies $\int_{F_N} f > (\int f) - \epsilon$.

□

2.13 Folland 2.17

Prove the following Proposition:

Proposition. 2.13:

Assume Fatou's lemma and deduce the monotone convergence theorem from it.

Proof. Let $\{f_n\}_1^\infty$ be a sequence in L^+ s.t. $f_j \leq f_{j+1} \ \forall j \in \mathbb{N}$, and $f = \lim_{n\to\infty} f_n$. If we're assuming Fatou's Lemma, then:

$$\int f = \int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

However, since $\{f_n\}_1^\infty$ is monotone increasing with the limit of f, we have $f_n \leq f \ \forall n \in \mathbb{N} \Rightarrow \int f_n \leq \int f \ \forall n \in \mathbb{N}$. And hence taking the $\limsup$ on both sides, we get:

$$\limsup_{n\to\infty} \int f_n \leq \limsup_{n\to\infty} \int f = \int f$$

Therefore, in combining these two inequalities, we see:

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n$$

Which can be true $\iff$ all the inequalities above are actually equalities, hence we have:

$$\lim_{n\to\infty} \int f_n = \limsup_{n\to\infty} \int f_n = \liminf_{n\to\infty} \int f_n = \int f$$

□

2.14 Differentiable functions are Borel Measurable

Exercise. 2.1:

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, show that its derivative f' is Borel Measurable.

Proof. Firstly, we note that by (Folland) Corollary 2.2, since $f \in C^1(\mathbb{R}) \Rightarrow f \in C(\mathbb{R})$, we have that f is Borel measurable.

Next, we prove that $g_n := f(x+1/n)$ is Borel measurable. This is actually quite easy since $h_n = x+1/n$ is naturally Borel measurable, and hence $f \circ h_n = g_n$ is Borel measurable since both f and g_n are Borel measurable.

Next, since $f \in C^1(\mathbb{R})$, we know that $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x),\ f'(x) \in \mathbb{R}$. Therefore, we can also say that $\lim_{n\to\infty} n(f(x+1/n) - f(x)) = f'(x)\ \forall x \in \mathbb{R}$. Since we already showed $f(x+1/n)$ and $f(x)$ are Borel Measurable, by (Folland) Proposition 2.6, $f'_n := n(f(x+1/n) - f(x))$ is Borel measurable $\forall n \in \mathbb{N}$. Finally, by (Folland) Proposition 2.7, we can conclude that $\underline{f'(x) = \lim_{n\to\infty} f'_n(x) \text{ is Borel measurable}}$ since $f \in C^1(\mathbb{R})$, $f'_n \to f$, and $\{f_n\}_1^\infty$ is a sequence of Borel measurable functions. $\square$

2.15 Folland 2.20

Prove the following Proposition:

Proposition. 2.14:

(A generalized Dominated Convergence Theorem) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e., $f_n \le g_n$ and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem).

Proof. By the same reasoning as in Folland, WLOG we may assume f_n and f are real-valued, and that $g_n + f_n \ge 0$ a.e., and $g_n - f_n \ge 0$ a.e. Now, we apply (Folland) Corollary (of Fatou's Lemma) 2.19 to both $g_n + f_n$ and $g_n - f_n$ as follows (we can do so due to the convergent and L^1 assumptions):

$$\int (g+f) = \int \lim(g_n + f_n) \le \liminf \int (g_n + f_n) = \int g + \liminf \int f_n$$

$$\int (g-f) = \int \lim(g_n - f_n) \le \liminf \int (g_n - f_n) = \int g - \limsup \int f_n$$

And so:

$$\limsup \int f_n - \int g \le -\int g + \int f \quad \text{and} \quad \int g + \int f \le \int g + \liminf \int f_n$$

And by combining these inequalities, we see that:

$$\limsup \int f_n \le \int f \le \liminf \int f_n$$

And since $f, f_n \in L^1$, we know that the above inequalities imply equalities, everywhere, I.e., $\lim \int f_n$ exists and $\int f_n \to \int f$. $\square$

2.16 Folland 2.21

Prove the following Proposition:

Proposition. 2.15:

Suppose $f_n, f \in L^1$ and $f_n \to f$ a.e. Then $\int |f_n - f| \to 0 \iff \int |f_n| \to \int |f|$, (Use (Folland) Exercise 20).

Proof. For the forward direction, assume $\int |f_n - f| \to 0$, then since:

$$\left| \int |f_n| - \int |f| \right| = \left| \int (|f_n| - |f|) \right| \leq \int \left| (|f_n| - |f|) \right| \leq \int |f_n - f|, \quad \text{since } |f_n| - |f| \leq |f_n - f|$$

we know that the right hand side $\to 0$ as $n \to \infty$, and since the above holds $\forall n \in \mathbb{N}$ (and since $f_n, f \in L^1$), $\left| \int |f_n| - \int |f| \right| \to 0 \Rightarrow \int |f_n| \to \int |f|$.

For the backward direction, assume $\int |f_n| \to \int |f|$. If we let $g_n := |f_n| + |f|$, then naturally $|f_n - f| \leq g_n$, and since $f_n, f \in L^1$, we know that $\int g_n = \int (|f_n| + |f|) = 2 \int |f|$. We may now invoke the generalized dominated convergence theorem, (Folland) Exercise 2.20 above, which implies:

$$\lim \int |f_n - f| = \int \lim |f_n - f|$$

And since $f_n \to f$, we therefore have $\int |f_n - f| \to 0$. □

2.17 Folland 2.24

2.18 Folland 2.34

Prove the following Proposition:

Proposition. 2.16:

Suppose $|f_n| \leq g \in L^1$ and $f_n \to f$ in measure.

a) $\int f = \lim \int f_n$.

b) $f_n \to f$ in L^1.

Before we begin, we present Folland Exercise 33 as a necessary Lemma for part a):

2.18.1 Folland 2.33

Lemma. 2.1:

If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Proof.

a) By (Folland) Theorem 2.30, $\exists$ a subsequence $\{f_{n_k}\}_1^\infty$ s.t. $f_{n_k} \to h$, where $f = h$ a.e. Furthermore, by (Folland) Proposition 2.11, since $f = h$ a.e., f is measurable. As is standard by this point, we may assume f_n and g are real-valued functions; therefore, $g + f_n \geq 0$ a.e., and $g - f_n \geq 0$ a.e. Moreover, we naturally have $g + f_n \to g + f$, $g - f_n \to g - f$ in measure. We now make use of our Lemma as follows:

$$\int g + \int f = \int (g + f) \leq \liminf \int (g + f_n) = \int g + \liminf \int f_n$$
$$\int g - \int f = \int (g - f) \leq \liminf \int (g - f_n) = \int g - \limsup \int f_n$$

And so in combining these inequalities, we have:

$$\limsup \int f_n \leq \int f \leq \liminf \int f_n$$

Which we recognize as in previous exercises to be true $\iff$ all the above inequalities are actually equalities; hence: $\int f = \lim \int f_n$.

b) From (Folland) Proposition 2.29, we know that since $f_n \to f$ in L^1, $f_n \to f$ in measure as well. Thus, since:

$$\mu\Big(\{x \in \mathcal{X} \mid ||f_n(x) - f(x)| - 0| \geq \epsilon\}\Big) = \mu\Big(\{x \in \mathcal{X} \mid |f_n(x) - f(x)| \geq \epsilon\}\Big) \to 0 \text{ as } n \to \infty$$

We also have that $|f_n - f| \to 0$ in measure. Furthermore, since: $|f_n - f| \leq |f_n| + |f| \leq 2g \in L^1$, we may apply Part a) to see that:

$$0 = \int 0 \; d\mu = \lim \int |f_n - f| d\mu$$

Hence $f_n \to f$ in L^1.

$\square$

2.19 Folland 2.39

Prove the following Proposition:

Proposition. 2.17:

If $f_n \to f$ almost uniformly, then $f_n \to f$ a.e. and in measure.

Proof. We first recall the $f_n \to f$ almost uniformly means that $\exists \{E_n\}_1^\infty \subset \mathcal{M}$ s.t. $\mu(E_n^c) < \frac{1}{n}$ and $f_n \to f$ uniformly on E_n. If we define $E := \cup_1^\infty E_n$, then $\mu(E^c) \leq \liminf \mu(E_n^c) = 0$; hence, $f_n \to f$ a.e.

Now to show $f_n \to f$ in measure, we proceed as follows. $\forall \epsilon, \delta > 0$, since $f_n \to f$ almost uniformly, $\exists E \in \mathcal{M}$ and an $N \in \mathbb{N}$ s.t. $\forall n \geq N, |f_n(x) - f(x)| < \epsilon \; \forall x \in E$ and $\mu(F^c) < \delta$. An immediate result of this set up is therefore:

$$\liminf_{n \to \infty} \mu\Big(\{x \in \mathcal{X} \mid |f_n(x) - f(x)| \geq \epsilon\}\Big) \leq \mu(F^c) < \delta$$

And since our result works $\forall \delta > 0$, letting $\delta \to 0$ proves $f_n \to f$ in measure.

$\square$

2.20 Folland 2.42

Prove the following Proposition:

Proposition. 2.18:

Let μ be a counting measure on $\mathbb{N}$. Then $f_n \to f$ in measure $\iff$ $f_n \to f$ uniformly.

Proof. For the forward direction, suppose $f_n \to f$ in measure (μ a counting measure on $\mathbb{N}$). Then $\forall \epsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$:

$$\mu\Big(\{x \in \mathbb{N} \mid |f_n(x) - f(x)| \geq \epsilon\}\Big) < \frac{1}{2} \quad \Rightarrow \quad \{x \in \mathbb{N} \mid |f_n(x) - f(x)| \geq \epsilon\} = \varnothing$$

I.e., $|f_n(x) - f(x)| < \epsilon \; \forall x \in \mathbb{N}$ (and $n \geq N$), which is by definition uniform convergence.

For the converse, suppose $f_n \to f$ uniformly (again μ a counting measure on $\mathbb{N}$). Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$:

$$|f_n(x) - f(x)| < \epsilon \; \forall x \in \mathbb{N} \quad \Rightarrow \quad \mu\Big(\{x \in \mathbb{N} \mid |f_n(x) - f(x)| \geq \epsilon\}\Big) = \mu(\varnothing) = 0$$

For which the latter equality vacuously satisfies our definition of convergence in measure. □

2.21 Folland 2.44: Lusin's Theorem

Prove the following Theorem:

Theorem. 2.1: Lusin's Theorem

If $f : [a.b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|E$ is continuous. (Use Egoroff's Theorem and (Folland) Theorem 2.26.)

Proof. Following the hint, by (Folland) Theorem 2.26, $\exists \{f_n\}_1^\infty$ s.t. $f_n : [a,b] \to \mathbb{C}$ and $f_n \to f$ a.e. Also, by Egoroff's Theorem, $\exists F \subset [a,b]$ s.t. $\mu(F^c) < \epsilon/2$ and $f_n \to f$ uniformly on F.

Naturally, since $f_n \to f$ uniformly on F^c, $f|F$ will be continuous. We now make use of (Folland) Theorem 1.18 which states that $\forall F \in \mathcal{M}_\mu$:

$$\mu(F) = \sup\{\mu(K) \mid K^{cpt} \subset F\}$$

And so by definition of sup, $\exists\, E^{cpt} \subset F$ s.t. $\mu(F \backslash E) < \epsilon/2$. We thus have:

$$\mu(E^c) = \mu(F^c) + \mu(F \backslash E) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

And since $E \subset F$, we know that $f|E$ is also continuous, thereby proving the Theorem. □

2.22 Folland 2.46

Prove the following Proposition:

Proposition. 2.19:

Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}, \mu =$ Lebesgue measure, and $\nu =$ counting measure. If $D = \{(x,x) \mid x \in [0,1]\}$ is the diagonal in $X \times Y$, then $\int\int \chi_D d\mu d\nu$, $\int\int \chi_D d\nu d\mu$, and $\int \chi_D d(\mu \times \nu)$ are all unequal. (To compute $\int \chi_D d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

Proof. We begin by making the following two observations: $\forall x \in [0,1]$, $\int \chi_D d\nu(y) = \int_{\{x\}} d\nu(y) = \nu(\{x\}) = 1$ and $\forall y \in [0,1]$, $\int \chi_D d\mu(x) = \int_{\{y\}} d\nu(x) = \mu(\{y\}) = 0$. Therefore, we'll now be able to compute $\int\int \chi_D d\mu d\nu$ and $\int\int \chi_D d\nu d\mu$ as follows:

$$\int\int \chi_D d\mu d\nu = \int \Big(\int \chi_D(x,y) d\mu(x)\Big) d\nu(y) = \int 0 d\nu(y) = 0$$

$$\int\int \chi_D d\nu d\mu = \int \Big(\int \chi_D(x,y) d\nu(y)\Big) d\mu(x) = \int_{[0,1]} d\mu(x) = 1$$

We now claim that $\int \chi_D d(\mu \times \nu) = \infty$. To see this, suppose $\{A_n \times B_n\}_1^\infty$ s.t. $A_n, B_n \subset [0,1]$ (measurable subsets) s.t. $D \subset \cup_1^\infty (A_n \times B_n)$. Therefore, $[0,1] \subset \cup_1^\infty (A_n \times B_n)$. Because of this, $\exists N \in \mathbb{N}$ s.t. $\mu^*(A_N \times B_N) > 0$, and explicitly $\mu(A_N) > 0$, and $\nu(B_N) = \infty$. Therefore, $\sum_1^\infty \mu(A_n)\nu(B_n) = \infty \Rightarrow \int \chi_D d(\mu \times \nu) = \infty$. □

2.23 Folland 2.48

Prove the following Proposition:

Proposition. 2.20:

Let $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$, $\mu = \nu =$ counting measure. Define:

$$f(m,n) = \begin{cases} 1 & \text{if } m = n \\ -1 & \text{if } m = n+1 \\ 0 & \text{otherwise} \end{cases}$$

Then $\int |f| d(\mu \times \nu) = \infty$, and $\int \int f \ d\mu d\nu$ and $\int \int f \ d\nu d\mu$ exist and are unequal.

Proof. We first claim that $\mu \times \mu$ is also a counting measure. We may actually note that fundamentally, a counting measure τ on $\mathbb{N} \times \mathbb{N}$ will satisfy $\tau(A \times B) = |A||B| = \mu(A)\nu(B)$. Therefore, since rectangles generate the product σ-algebra, the σ-finitness of τ implies $\mu \times \nu = \tau$.

We now proceed to computing each of the quantities of interest. For the first, if we let $E := \cup_1^\infty \{\{(n,n)\} \cup \{(n, n+1)\}\}$, then clearly $|E| = \infty$, and $|f| = \chi_E$. Therefore:

$$\int |f| d(\mu \times \nu) = |E| = \infty$$

Furthermore, the other two calculations are nearly immediate:

$$\int \int f d\mu d\nu = \int \int f(m,n) d\mu(m) d\nu(n) = \sum_n \sum_m f(m,n) = \sum_n 0 = 0$$

$$\int \int f d\mu d\nu = \int \int f(m,n) d\nu(n) d\nu(m) = \sum_m \sum_n f(m,n) = \sum_m \chi_{\{m=1\}} = 1$$

□

2.24 Folland 2.49

Prove the following Proposition:

Proposition. 2.21:

Prove (Folland) Theorem 2.38 by using Theorem 2.37 and Proposition 2.12 together with the following lemmas:

a) If $E \in \mathcal{M} \times \mathcal{N}$ and $\mu \times \nu(E) = 0$, then $\nu(E_x) = \mu(E^y) = 0$ for a.e. x and y.

b) If f is $\mathcal{L}$-measurable and $f = 0$ λ-a.e., then f_x and f^y are integrable for a.e. x and y, and $\int f_x d\nu = \int f^y d\mu = 0$ for a.e. x and y. (Here the completeness of μ and ν is needed.)

Proof.

a) Immediately by (Folland) Theorem 2.36, since $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite measure spaces, we have:

$$0 = \mu \times \nu(E) = \int \nu(E_x) d\mu(x) = \int \mu(E^y) d\nu(y)$$

b) Let us define $F := \{(x, y) \in \mathcal{M} \times \mathcal{N} \mid f(x, y) \neq 0\}$. Thus, $\exists E \in M \otimes N$ where $\mu \times \nu(E) = 0$ and $F \subset E$. From a), $\mu(E_x) = \nu(E^y) = 0$ for x, y a.e. Furthermore, by the fact that $F_x \subset E_x$ and $F^y \subset E^y$, by linearity of measures we have $\mu(F_x) = \nu(F^y) = 0$ as well. We may now conclude thus that:
$$\int |f_x| d\nu = \int \chi_{F_x} |f_x| d\nu = 0 = \int \chi_{F^y} |f^y| d\mu = \int |f^y| d\mu = 0 \quad \text{for a.e. } x \text{ and } y$$
For which the intended result trivially follows.

Please see the next (attached) page for our proof of (Folland) Theorem 2.38. □

3 Chapter 3

3.1 Folland 3.2

Prove the following Proposition:

Proposition. 3.1:

a) If ν is a signed measure, E is ν-null $\iff |\nu|(E) = 0$.

b) If ν ane μ are signed measures, $\nu \perp \mu \iff |\nu| \perp \mu \iff \nu^+ \perp \mu$ and $\nu^- \perp \mu$.

Proof.

a) For the forward direction, suppose ν is a signed measure and that E is ν-null. Suppose for the sake of contradiction that $|\nu|(E) = \nu^+(E) + \nu^-(E) > 0$, where $\nu = \nu^+ - \nu^-$ is the Jordon Decomposition of ν. By the Hahn Decomposition Theorem, $\exists\ P, N$ s.t. $\nu(X) = \nu(P \cup N) = \nu^+(P) - \nu^-(N)$ and $\nu^+(N) = 0 = \nu^-(P)$.

We thus can thus make the following observations:
$$|\nu|(E) = \nu^+(E) + \nu^-(E) = 2\nu^+(E) > 0 \quad \text{since } \nu(E) = 0 \Rightarrow \nu^+(E) = \nu^-(E)$$
$$\nu^+(E \cap P) = \nu^+(E \cap N) + \nu^+(E \cap P) = \nu^+(E \cap X) = \nu^+(E) > 0 \quad \text{since } 2\nu^+(E) > 0$$
$$\nu^-(E \cap P) \leq \nu^-(P) = 0$$
And so:
$$\nu(E \cap P) = \nu^+(E \cap P) + \nu^-(E \cap P) = \nu^+(E \cap P) > 0$$
However, since $E \cap P \subset E$, and we are assuming that $\nu(E) = 0$, we arrive at a contradiction with the last inequality. Thus, actually if E is ν-null, $|\nu|(E) = 0$.

For the converse, suppose $|\nu|(E) = 0$, hence $|\nu|(E') = 0\ \forall E' \subset E$ (E' measurable). Since $|\nu|(E') = \nu^+(E') + \nu^-(E') = 0 \iff \nu^+(E') = 0 = \nu^-(E')$, we thus trivially satisfy $\nu(E') = \nu^+(E') - \nu^-(E') = 0$ since both are already zero.

b) Let us recall that, explicitly, if $\exists\ E, F \in \mathcal{M}$ s.t. $E \cap F = \varnothing$, $E \sqcup F = X$ and $\mu(E') = 0 = \nu(F')\ \forall E' \subset E, F' \subset F$ (E', F' measurable), we denote this property as $\nu \perp \mu$.

We begin by showing $\nu \perp \mu \Rightarrow |\nu| \perp \mu$. To see this, since F is ν-null, by Part a), we know that $|\nu|(F) = 0$. Since $|\nu|$ is a positive (regular) measure, by monotonicity we have that $|\nu|$ is F-null. Thus, by definition, $|\nu| \perp \mu$.

We now show $|\nu| \perp \mu \Rightarrow \nu^+ \perp \mu$ and $\nu^- \perp \mu$. Since $\nu = \nu^+ + \nu^-$, we have $\nu^+ \leq \nu$ and $\nu^- \leq \nu$, and so $\nu^+(F') = \nu^-(F') = 0$, I.e., F is both ν^+-null and ν^--null; hence, $\nu^+ \perp \mu$ and $\nu^- \perp \mu$.

We may now complete Part b) by showing $[\nu^+ \perp \mu$ and $\nu^- \perp \mu] \Rightarrow \nu \perp \mu$. Let us make explicit the properties associated with $\nu^+ \perp \mu$ and $\nu^- \perp \mu$ by replacing the roles of E, F (from the beginning of this proof) with A_1, A_2 for $\nu^+ \perp \mu$ and B_1, B_2 for $\nu^- \perp \mu$. We first note that since A_1, B_1 are both μ-null, so too is $A_1 \cup B_1$. This is true since by looking at the following representation: $A_1 \cup B_1 \equiv A_1 \sqcup (B_1 \backslash A_1)$, and in noting $B_1 \backslash A_1 \subset B_1$, $\forall E' \subset A_1 \cup B_1$, $\exists A_1' \subset A_1, B_1' \subset B_1$ s.t. $E' = B_1' \sqcup A_1$ and both B_1' and A_1' are μ-null. Furthermore, since $A_1 \cup A_2 = X = B_1 \cup B_2$, we have $X \backslash (A_1 \cup B_1) = A_2 \cap B_2$, which is both ν^+-null and ν^--null since $A_2 \cap B_2 \subset A_2$ and $A_2 \cap B_2 \subset B_2$. Thus, by setting $E = A_1 \cup B_1$, and $F = A_2 \cap B_2$, we see that indeed $\nu \perp \mu$.

$\square$

3.2 Folland 3.7

Prove the following Proposition:

Proposition. 3.2:

Suppose that ν is a signed measure on $(X, \mathcal{M})$ and $E \in \mathcal{M}$.

a) $\nu^+(E) = \sup\left\{\nu(F) \;\middle|\; E \in \mathcal{M}, F \subset E\right\}$ and $\nu^-(E) = -\inf\left\{\nu(F) \;\middle|\; F \in \mathcal{M}, F \subset E\right\}$.

b) $|\nu|(E) = \sup\left\{ \sum_1^n |\nu(E_j)| \;\middle|\; n \in \mathbb{N}, E_1, \ldots, E_n \text{ are disjoint, and } \sqcup_1^n E_j = E\right\}$.

Proof.

a) For the first equality, $\forall F \subset E$ we have:

$$\nu(F) = \nu^+(F) - \nu^-(F) \leq \nu^+(F) \leq \nu^+(E)$$

And so $\nu^+(E) \geq \sup\{\nu(F) \mid E \in \mathcal{M}, F \subset E\}$. To see the reverse inequality, if P, N are our Hahn Decomposition of ν, we naturally have $\nu^+(E) = \nu(E \cap P)$, and since $E \cap P \subset E$, $\nu^+(E) \leq \sup\{\nu(F) \mid E \in \mathcal{M}, F \subset E\}$, and so:

$$\nu^+(E) = \sup\left\{\nu(F) \;\middle|\; E \in \mathcal{M}, F \subset E\right\}$$

For the second inequality, this follows very similarly. Explicitly, $\forall F \subset E$, we have:

$$-\nu(F) = \nu^-(F) - \nu^+(F) \leq \nu^-(F) \leq \nu^-(E)$$

And so $\nu^-(F) \geq \sup\{-\nu(F) \mid F \in \mathcal{M}, F \subset E\} = -\inf\{\nu(F) \mid F \in \mathcal{M}, F \subset E\}$. For the reverse inequality, since $\nu^-(E) = -\nu(E \cap N)$, and since $E \cap N \subset E$, $\nu^-(E) \leq \sup\{-\nu(F) \mid F \in \mathcal{M}, F \subset E\} = -\inf\{\nu(F) \mid F \in \mathcal{M}, F \subset E\}$. Combining our two inequalities, we see:

$$\nu^-(E) = -\inf\left\{\nu(F) \;\middle|\; F \in \mathcal{M}, F \subset E\right\}$$

b) Firstly, if P, N are again the Hahn Decomposition of ν, then $E = (E \cap N) \sqcup (E \cap P)$, and so:

$$\begin{aligned}
|\nu|(E) = \nu^+(E) + \nu^-(E) &= \nu^+(E \cap P) + \nu^-(E \cap N) \\
&= \nu^+(E \cap P) + \nu^-(E \cap N) + \underbrace{\nu^+(E \cap N) + \nu^-(E \cap P)}_{=0} \\
&= |\nu(E \cap P)| + |\nu(E \cap N)|
\end{aligned}$$

And so:

$$|\nu|(E) \leq \sup\left\{\sum_1^n |\nu(E_j)| \,\middle|\, n \in \mathbb{N}, E_1, \dots, E_n \text{ are disjoint, and } \bigsqcup_1^n E_j = E\right\}$$

To see the reverse inequality, we note that $\forall E = \sqcup_1^n E_j$, we have:

$$\begin{aligned}
\sum_{j=1}^n |\nu(E_j)| &= \sum_{j=1}^n |\nu^+(E_j) - \nu^-(E_j)| \\
&\leq \sum_{j=1}^n \big(\nu^+(E_j) + \nu^-(E_j)\big) \\
&= \sum_{j=1}^n |\nu|(E_j) \\
&= |\nu|\left(\bigsqcup_{j=1}^n E_j\right) = |\nu|(E)
\end{aligned}$$

And so:

$$|\nu|(E) \geq \sup\left\{\sum_1^n |\nu(E_j)| \,\middle|\, n \in \mathbb{N}, E_1, \dots, E_n \text{ are disjoint, and } \bigsqcup_1^n E_j = E\right\}$$

And hence combining the two inequalities, we have:

$$|\nu|(E) = \sup\left\{\sum_1^n |\nu(E_j)| \,\middle|\, n \in \mathbb{N}, E_1, \dots, E_n \text{ are disjoint, and } \bigsqcup_1^n E_j = E\right\}$$

□

3.3 Folland 3.12

Prove the following Proposition:

Proposition. 3.3:

For $j = 1, 2$, let μ_j, ν_j be σ-finite measures on $(X_j, \mathcal{M}_j)$ s.t. $\nu_j << \mu_j$. Then $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$ and:

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = \frac{d\nu_1}{d\mu_1}(x_1)\frac{d\nu_2}{d\mu_2}(x_2)$$

Proof. Let us begin by defining $f_j := \frac{d\nu_j}{d\mu_j}$ for $j = 1, 2$. Thus, if $A_1 \times A_2$ is measurable, by the definition of product measure and Radon-Nikodym derivative, we have:

$$\begin{aligned}
\nu_1 \times \nu_2(A_1 \times A_2) = \nu_1(A_1)\nu_2(A_2) &= \int_{A_1} f_1 d\mu_1 \int_{A_2} f_2 d\mu_2 \\
&= \int f_1 \chi_{A_1} d\mu_1 \int f_2 \chi_{A_2} d\mu_2 \\
&= \int\int f_1 f_2 \chi_{A_1}\chi_{A_2} d\mu_1 d\mu_2 \\
&\overset{*}{=} \int\int_{A_1 \times A_2} f_1 f_2 d(\mu_1 \times \mu_2)
\end{aligned}$$

Where we have $\stackrel{*}{=}$ by Tonelli's Theorem. Therefore, on $A_1 \times A_2$ measurable, $(f_1 f_2)(\mu_1 \mu_2) = \nu_1 \nu_2$; and thus we also have equality on the algebra of finite unions of $A_1 \times A_2$'s. Furhermore, by the uniqueness of the extension from premeasure to measure, $(f_1 f_2)(\mu_1 \mu_2) = \nu_1 \nu_2$ on $\mathcal{M}_1 \otimes \mathcal{M}_2$. We thus immediately have that if $(\mu_1 \times \mu_2)(E) = 0 \;\Rightarrow\; (\nu_1 \times \nu_2)(E) = 0$, and so $\nu_1 \times \nu_2 << \mu_1 \times \mu_1$. Finally, since the Radon-Nikodym derivative is unique, we have:

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x_1, x_2) = f_1(x_1) f_2(x_2) = \frac{d\nu_1}{d\mu_1}(x_1) \frac{d\nu_2}{d\mu_2}(x_2)$$

□

3.4 Folland 3.13

Prove the following Proposition:

Proposition. 3.4:

Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$, then:

a) $m << \mu$ but $dm \neq f d\mu$ for any f.

b) μ has no Lebesgue decomposition with respect to m.

Proof.

a) Firstly, if $E \in \mathcal{M}$ and $\mu(E) = 0$, then it must be that $E = \varnothing$, and so $m(E) = m(\varnothing) = 0$; I.e., $m << \mu$. Suppose for the sake of contradiction that $dm = f d\mu$, then $\forall x \in [0,1]$ and $E = \{x\}$, we have:

$$0 = m(E) = \int_E f d\mu = \int_E dm = m(E) = 0$$

Thus we must have that $f \equiv 0$ on $[0,1]$. However:

$$1 = m([0,1]) = \int_{[0,1]} f d\mu = \int_{[0,1]} 0 d\mu = 0$$

I.e. we've reached a contradiction and hence $dm \neq f d\mu$ for any f.

b) Suppose, for the sake of contradiction, that μ has a Lebesgue decomposition w.r.t. m; namely: $\mu = \lambda + \rho$ where $\lambda \perp m$ and $\rho << m$. Since $\lambda \perp m$, by definition we know that $\exists E, F$ s.t $X = E \sqcup F$ where E is λ-null and F is m-null (or just $m(F) = 0$ since m is a positive measure). Suppose $x \in F$, then $\mu(\{x\}) = 1$, but $\lambda(\{x\}) = 0$ and $m(\{x\}) = 0 \Rightarrow \rho(\{x\}) = 0$, which would be a contradiction unless we have $F = \varnothing$. Thus, since $X = E \sqcup F = E \sqcup \varnothing \Rightarrow E = X$. However, since $m(E) = m([0,1]) = 1$, yet we are requiring E to be m-null, we arrive at a contradiction. Thus, $\nexists$ a Lebesgue decomposition of μ w.r.t. m.

□

3.5 Folland 3.17

Prove the following Proposition:

Proposition. 3.5:

Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|\mathcal{N}$. If $f \in L^1(\mu)$, $\exists g \in L^1(\nu)$ (thus g is $\mathcal{N}$-measurable) s.t. $\int_E f d\mu = \int_E g d\nu$ $\forall E \in \mathcal{N}$; if g' is another such function, then $g = g'$ ν-a.e. (In Probability Theory, g is call the **conditional expectation** of f on $\mathcal{N}$.)

Proof. Let us begin by defining the measure λ s.t. $d\lambda = fd\mu$ and its integration is restricted to $E \in \mathcal{N}$; I.e., $\forall E \in \mathcal{N}$, we have $\lambda(E) = \int_E fd\mu$. To easily see that $\lambda << \nu$, note that if $\nu(E) = 0 \Rightarrow \mu(E) = 0 \Rightarrow \lambda(E) = \int_E dd\mu = 0$. We thus have shown the necessary conditions for us to invoke The Lebesgue-Radon-Nikodym theorem. Explicitly, the Radon-Nikodym derivative, $g = \frac{d\lambda}{d\nu}$ exists and is ν-integrable where $fd\mu = d\lambda = gd\nu$; I.e.;

$$\int_E fd\mu = \int_E d\lambda = \int_E gd\nu \quad \forall E \in \mathcal{N}$$

Finally, if g' satisfies $\int_E fd\mu = \int_E g'd\nu$, then naturally $d\lambda = g'd\nu$, and since the Radon-Nikodym derivative is unique, we must also have $g = g'$ ν-a.e. □

3.6 Folland 3.20

Prove the following Proposition:

Proposition. 3.6:

If ν is a complex measure on $(X, \mathcal{M})$ and $\nu(X) = |\nu|(X)$, then $\nu = |\nu|$.

Proof. Suppose that $d|\nu| = fd\mu$ as in the definition of $|\nu|$. Then if $E \in \mathcal{M}$, then we will have:

$$\nu(E) + \nu(E^c) = \nu(X) \overset{*}{=} |\nu|(X) = |\nu|(E) + |\nu|(E^c), \quad \text{where we have } \overset{*}{=} \text{ by assumption}$$

And so:

$$\nu(E^c) - |\nu|(E^c) = |\nu|(E) - \nu(E)$$

Taking the real part of the LHS, and using (Folland) Proposition 3.13a ($|\nu(E)| \leq |\nu|(E)$), we see that:

$$\begin{aligned} Re\big(\nu(E^c) - |\nu|(E^c)\big) &\leq Re(\nu(E^c) - |\nu(E^c)|) \\ &= \nu_r(E^c) - |\nu(E^c)| \\ &= \nu_r(E^c) - \sqrt{\nu_r^2(E^c) + \nu_i^2(E^c)} \\ &\leq \nu_r(E^c) - \sqrt{\nu_r^2(E^c)} = 0 \end{aligned}$$

And similarly for the RHS:

$$\begin{aligned} Re\big(|\nu|(E) - \nu(E)\big) &\geq Re(|\nu(E)| - \nu(E)) \\ &= |\nu(E)| - \nu_r(E) \\ &= \sqrt{\nu_r^2(E) + \nu_i^2(E)} - \nu_r(E^c) \\ &\leq \sqrt{\nu_r^2(E)} - \nu_r(E) = 0 \end{aligned}$$

And so combining the fact that $Re(LHS) \leq 0 \leq Re(RHS)$, but obviously since $Re(LHS) = Re(RHS)$, we must have that:

$$Re(|\nu|(E) - \nu(E)) = 0 \quad \Rightarrow \quad |\nu|(E) = \nu_r(E)$$

But since, again by (Folland) Proposition 3.13a, we have $|\nu(E)| \leq |\nu|(E)$, we see that this must be true $\iff \nu_i(E) = 0$, and so:

$$|\nu|(E) = \nu_r(E) = \nu_r(E) + \nu_i(E) = \nu(E) \quad \forall E \in \mathcal{M} \quad (\text{I.e.}, \nu = |\nu|)$$

□

3.7 Folland 3.21

Prove the following Proposition:

Proposition. 3.7:

Let ν be a complex measure on $(X, \mathcal{M})$. If $E \in \mathcal{M}$, define:

$$\mu_1(E) = \sup\left\{\sum_{j=1}^n |\nu(E_j)| \;\middle|\; n \in \mathbb{N}, E_1, \ldots, E_n \text{ disjoint}, E = \sqcup_{j=1}^n E_j\right\}$$

$$\mu_2(E) = \sup\left\{\sum_{j=1}^\infty |\nu(E_j)| \;\middle|\; n \in \mathbb{N}, E_1, E_2, \ldots \text{ disjoint}, E = \sqcup_{j=1}^\infty E_j\right\}$$

$$\mu_3(E) = \sup\left\{\left|\int_E f d\nu\right| \;\middle|\; |f| \leq 1\right\}$$

Then $\mu_1 = \mu_2 = \mu_3 = |\nu|$. (First show that $\mu_1 \leq \mu_2 \leq \mu_3$. To see that $\mu_3 = |\nu|$, let $f = \frac{d\nu}{d|\nu|}$ and apply (Folland) Prop 3.13. To see that $\mu_3 \leq \mu_1$, approximate f by a simple function.)

Proof. We proceed as in the hint. Trivially, $\mu_1 \leq \mu_2$ since given $E \in \mathcal{M}$, $\left\{\{E_j\}_1^\infty \mid E = \sqcup_1^\infty E_j\right\} \supset \left\{\{E_j\}_1^n \mid E = \sqcup_1^n E_j\right\}$ (set $E_j = \varnothing \; \forall j > n$).

To see $\mu_2 \leq |\nu| \leq \mu_3$, let $f := \frac{d\nu}{d|\nu|}$, and so if $E = \sqcup_1^\infty E_j$, we have:

$$\begin{aligned}
\left[\sum_{j=1}^\infty |\nu(E_j)|\right] &\leq \sum_{j=1}^\infty |\nu|(E_j) && \text{by (Folland) Prop. 3.13a}\\
&= \Big[|\nu|(E)\Big] = \int_E d|\nu| \\
&= \int_E |f|^2 d|\nu| && \text{by (Folland) Prop. 3.13b}\\
&= \int_E \overline{f} f d|\nu| \\
&= \int_E \overline{f} d\nu && \text{by (Folland) Prop. 3.9a}\\
&\leq \left|\int_E \overline{f} d\nu\right| \in \left[\left\{\left\{\int_E f d\nu\right\} \;\middle|\; |f| \leq 1\right\}\right]
\end{aligned}$$

And so $\mu_2 \leq |\nu| \leq \mu_3$ by the steps with square brackets around them.

To show now that $\mu_3 \leq \mu_1$, let $\overline{\mathbb{D}} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. Trivially $\overline{\mathbb{D}}$ is compact, and thus $\exists \{z_j\}_1^n$ s.t. $\forall \epsilon > 0$:

$$\bigcup_1^n B_\epsilon(z_j) \supset \overline{\mathbb{D}}$$

Moreover, by definition of supremum, $\forall \epsilon > 0$, $\exists f$ s.t. $|f| \leq 1$ and:

$$\mu_3(E) \leq \left|\int_E f d\nu\right| + \epsilon$$

If we are assuming $|f| \leq 1$, then $f^{-1}(\overline{\mathbb{D}}) = X$, and so we will have $X = \cup_1^n f^{-1}(B_\epsilon(z_j))$ as well. By defining $B_j := f^{-1}(B_\epsilon(z_j))$, we now perform the standard "shuffle" to make a disjoint sequence $\{A_j\}_1^n$

out of $\{B_j\}_1^n$, namely we let $A_1 = B_1$, and $A_j = B_j \setminus \cup_{i=1}^{j} B_i$, so that $X = \sqcup_1^n A_j$. Now, in following the hint, we explicitly define the simple function $\phi := \sum_1^n z_j \chi_{A_j}$.
Naturally $|\phi| \leq 1$ and $|f(x) - \phi(x)| < \epsilon$. Thus:

$$\begin{aligned}
\mu_3(E) \leq \left| \int_E f d\nu \right| + \epsilon &\leq \left| \int_E f d\nu \right| - \left| \int_E \phi d\nu \right| + \left| \int_E \phi d\nu \right| + \epsilon \\
&\leq \left| \int_E f d\nu - \int_E \phi d\nu \right| + \left| \int_E \phi d\nu \right| + \epsilon \qquad \text{by the Reverse Triangle Inequality} \\
&\leq \left| \int_E |f - \phi| d\nu \right| + \left| \int_E \phi d\nu \right| + \epsilon \\
&\leq \int_E \epsilon d|\nu| + \left| \int_E \phi d\nu \right| + \epsilon \\
&\leq \epsilon|\nu|(E) + \epsilon + \left| \int_E \phi d\nu \right|
\end{aligned}$$

So, by letting $\epsilon \to 0$, we have $\mu_3(E) \leq |\int_E \phi d|\nu||$. Now, let us define $\{E_j\}_1^n$ by $E_j = A_j \cap E$ so that $E = \sqcup_1^n E_j$; thus:

$$\begin{aligned}
\mu_3(E) \leq \left| \int_E \phi d\nu \right| = \left| \int \sum_{j=1}^{n} z_j \chi_{A_j} \chi_E d\nu \right| &= \left| \sum_{j=1}^{n} z_j \int_{E_j} d\nu \right| \\
&= \left| \sum_{j=1}^{n} \nu(E_j) \right| \leq \sum_{j=1}^{n} |z_j||\nu(E_j)| \\
&\leq \sum_{j=1}^{n} |\nu(E_j)| \in \left\{ \{E_j\}_1^n \mid E = \sqcup_1^n E_j \right\}
\end{aligned}$$

And so $\mu_3 \leq \mu_1$. Thus since we were able to show $\mu_1 \leq \mu_2 \leq |\nu| \leq \mu_3 \leq \mu_1$, every inequality above is actually an equality and in fact: $\mu_1 = \mu_2 = \mu_3 = |\nu|$.

□

3.8 Folland 3.24

Prove the following Proposition:

Proposition. 3.8:

If $f \in L^1_{\text{loc}}$ and f is continuous at x, then x is in the Lebesgue set of f.

Proof. To show that x is in the Lebesgue set of f, we need to show that:

$$\lim_{r \searrow 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| dy = 0$$

To see this, suppose that $\epsilon > 0$. By the definition of continuity of f at x, we know that $\exists \delta > 0$ s.t. if $||x - y|| < \delta$, I.e., $y \in B_\delta(x)$, we have $|f(x) - f(y)| < \epsilon$. We therefore yield the following inequality for $0 < r < \delta$:

$$\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \leq \frac{1}{m(B_r(x))} \int_{B_r(x)} \epsilon dy = A_r(\epsilon) = \epsilon$$

Thus, since ϵ was arbitrary, we may conclude that our limit is indeed $= 0$; I.e., x is in the Lebesgue set of f. □

3.9 Folland 3.25

Prove the following Proposition:

Proposition. 3.9:

If E is a Borel set in $\mathbb{R}^n$, the **density**, $D_E(x)$, of x is defined as:

$$D_E(x) = \lim_{r \searrow 0} \frac{m(E \cap B_r(x))}{m(B_r(x))}$$

whenever the limit exists.

a) Show that:

$$D_E(x) = \begin{cases} 1 & \text{for a.e. } x \in E \\ 0 & \text{for a.e. } x \in E^c \end{cases}$$

b) Find examples of E and x s.t. $D_E(x)$ is a given number $\alpha \in (0,1)$, or such that $D_E(x)$ does not exist.

Proof.

a) Let us begin by defining $\nu(A) := m(E \cap A)\ \forall A \in \mathcal{B}_{\mathbb{R}^n}$. Then, by construction we have $\nu << m$ and $\frac{d\nu}{dm} = \chi_E$. Furthermore, since $\{B_r(x)\}_{r>0}$ vacuously satisfies the requirements for a set to shrink nicely to $x \in \mathbb{R}^n$, we may make use of (Folland) Theorem 3.22. Explicitly, by 3.22, we have:

$$D_E(x) = \lim_{r \searrow 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} = \lim_{r \searrow 0} \frac{\nu(B_r(x))}{m(B_r(x))} = \chi_E \quad \text{for } m - \text{almost every} x \in \mathbb{R}^n$$

Which is precisely what we wanted to show.

b) For the first example, we are looking for an E and an x s.t. $D_E(x) = \alpha$, where $\alpha \in (0,1)$. Suppose we are dealing in $\mathcal{B}_{\mathbb{R}^2}$ and we set $x = (0,0)$ and let $E = \{(x,y) \mid x = t\cos(\theta), y = t\sin(\theta), t > 0, \theta \in (0, 2\pi\alpha)\}$. Intuitively, E is the interior of the 2-dimensional, infinitely-extending, cone whose vertex is the point x, and walls are defined as the positive x-axis and the line starting at the origin and passing thought the point $(\cos(2\pi\alpha), \sin(2\pi\alpha))$ on the unit circle. Therefore, $E \cap B_r(x)$ will be the interior of the same cone as defined before, but now bounded by the curve beginning at $(r, 0)$, which traverses c.c.w. along $C_r(0)$ and stops at $(r\cos(2\pi\alpha), r\sin(2\pi\alpha))$. With this geometrical understanding, we can now easily recognize that since $m(B_r(x)) = 2\pi r^2$, $m(E \cap B_r(x)) = \alpha m(B_r(x)) = \alpha 2\pi r^2$. Since our results are true $\forall r > 0$, we have:

$$D_E(x) = \lim_{r \searrow 0} \frac{m(E \cap B(r,x))}{m(B(r,x))} = \lim_{r \searrow 0} \frac{\alpha 2\pi r^2}{2\pi r^2} = \alpha$$

For the second example, we are looking for an E and an x s.t. $D_E(x)$ does not exist. Suppose for this example we turn our thoughts to $\mathcal{B}_{\mathbb{R}^1}$ (so that $B_r(x) = (x - r, x + r)$). Let us now set $x = 0$, and define E as follows:

$$E = \bigsqcup_{n=1}^{\infty} \left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}} \right) = \left(\frac{1}{8}, \frac{1}{4} \right) \sqcup \left(\frac{1}{32}, \frac{1}{16} \right) \sqcup \left(\frac{1}{128}, \frac{1}{64} \right) \sqcup \cdots$$

Our strategy henceforth will be to compose a countable subsequence, r_k, s.t. $r_k \searrow 0$ and where $\lim_{k\to\infty} \frac{m(E \cap B_{r_k}(x))}{m(B_r(x))}$ will be undefined, therefore also rendering $D_E(x)$ undefined. To do this, we set $r_k = \frac{1}{2^k}$. Naturally, we have $m(B_{r_k}(x)) = \frac{2}{2^k} = \frac{1}{2^{k-1}}$. We now decompose (with a slight abuse of notation) $\{r_k\}_1^\infty = \{r_{2l}\}_1^\infty +' \{r_{2l+1}\}_1^\infty$, I.e. separate r_k into two subsequences, one where k is even, the other when k is odd. For the former (k is even), we have:

$$m(E \cap B_{r_k(x)}) = \sum_{n \geq l} m\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}\right) = \sum_{n \geq l} \frac{1}{2^{2n+1}} = \frac{1}{2^k} \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \left(\frac{1}{2^k}\right)\left(\frac{2}{3}\right) = \frac{1}{3 \cdot 2^{k-1}}$$

And (quite) similarly for k being odd we have:

$$m(E \cap B_{r_k(x)}) = \sum_{n \geq l+1} m\left(\frac{1}{2^{2n+1}}, \frac{1}{2^{2n}}\right) = \sum_{n \geq l+1} \frac{1}{2^{2n+1}} = \frac{1}{2^{k+1}} \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \left(\frac{1}{2^{k+1}}\right)\left(\frac{2}{3}\right) = \frac{1}{3 \cdot 2^k}$$

And so, in recalling again that $m(B_{r_k}) = \frac{1}{2^{k-1}}$ we have:

$$\frac{m(E \cap B_{r_k}(x))}{m(B_{r_k}(x))} = \begin{cases} \frac{1/(3 \cdot 2^{k-1})}{1/2^{k-1}} = \frac{1}{3} & \text{if } k = 2l, l \in \mathbb{N} \\ \frac{1/(3 \cdot 2^k)}{1/2^{k-1}} = \frac{1}{6} & \text{if } k = 2l+1, l \in \mathbb{N} \end{cases}$$

And so $\lim_{k \to \infty} \frac{m(E \cap B_{r_k}(x))}{B_{r_k}(x)}$ is undefined, and therefore so too is $D_E(x)$ by our previous reasoning.

□

3.10 Folland 3.26

Prove the following Proposition:

Proposition. 3.10:

If λ and μ are positive, mutually singular Borel measures on $\mathbb{R}^n$ and $\lambda + \mu$ is regular, then so are λ and μ.

Proof. If $K^{cpt} \subset \mathcal{B}_{\mathbb{R}^n}$, then $\lambda(K), \mu(K) < (\lambda + \nu)(K) < \infty$. Furthermore, suppose that E, F form the singular decomposition of λ, μ; I.e., $\mathbb{R}^n = E \sqcup F$ and $\forall F_1 \subset F, F_1 \in \mathcal{B}_{\mathbb{R}^n}$, $\mu(F_1) = 0$, and similarly for E w.r.t. λ.

Suppose now that $A \in \mathcal{B}_{\mathbb{R}^n}$. By definition of $\lambda + \mu$'s regularity, we know that:

$$(\lambda + \mu)(A) = \inf \left\{(\lambda + \mu)(\mathcal{U}^{open}) \mid \mathcal{U} \supset E\right\}$$

Therefore, $\forall \epsilon = 2^{-k}, k \in \mathbb{N}$, $\exists\ \mathcal{U}^{open}$ s.t. $(\lambda + \mu)(\mathcal{U}_k) < (\lambda + \mu)(A) + \epsilon$. Thus, we may construct a countable sequence of these such $\mathcal{U}_k$'s, namely $\{\mathcal{U}_k\}_1^\infty$, for which when letting $k \to \infty$, we have:

$$\lim_{k \to \infty} (\lambda + \mu)(\mathcal{U}_k) = (\lambda + \mu)(A), \quad \text{where } (\lambda + \mu)(\mathcal{U}_k) \geq (\lambda + \mu)(A)\ \forall k \in \mathbb{N} \text{ by positivity of measures}$$

By our set up of the singular decomposition of λ, μ we also note that we may express $(\lambda + \mu)(\mathcal{U}_k) = (\lambda + \mu)(\mathcal{U}_k \cap E) + (\lambda + \mu)(\mathcal{U}_k \cap F) = \mu(\mathcal{U}_k \cap E) + \lambda(\mathcal{U}_k \cap F)$, and similarly for A, namely: $(\lambda + \mu)(A) = \mu(A \cap E) + \lambda(A \cap F)$. Furthermore, since μ, λ are positive measures, and by construction $\mathcal{U}_k \supset A \Rightarrow \mathcal{U}_k \cap E \supset A \cap E$ (and similarly $\mathcal{U}_k \cap F \supset A \cap F$), we have $\mu(\mathcal{U}_k \cap E) \geq \mu(A \cap E)$ and $\lambda(\mathcal{U}_k \cap F) \geq \lambda(A \cap F)$. By applying the last result twice, we can reach the following result:

$$\begin{aligned}
(\lambda + \mu)(\mathcal{U}_k) - (\lambda + \mu)(A) &= \lambda(\mathcal{U}_k \cap F) + \mu(\mathcal{U}_k \cap E) - \lambda(A \cap F) - \mu(A \cap E) \\
&\geq \lambda(\mathcal{U}_k \cap F) - \lambda(A \cap F) \\
&= \lambda(\mathcal{U}_k \cap F) + \underbrace{\lambda(\mathcal{U}_k \cap E)}_{=0} - \lambda(A \cap F) - \underbrace{\lambda(A \cap E)}_{=0} \\
&= \lambda(\mathcal{U}_k) - \lambda(A) \geq 0
\end{aligned}$$

For which we already showed that the LHS has limit $= 0$, and thus taking limits on every equation in the above reasoning shows that $\lim_{k\to\infty} \lambda(\mathcal{U}_k) = \lambda(A)$. Furthermore, by the exact same steps but in swapping $\lambda \leftrightarrow \mu$, we see that $\lim_{k\to\infty} \mu(\mathcal{U}_k) = \mu(A)$ as well. Therefore, the same approximation by open sets from above for the definition of $(\lambda + \mu)$'s regularity also works as an approximation by open sets from above for all sets $A \in \mathcal{B}_{\mathbb{R}^n}$ for λ and μ, hence we have arrived at the definition of λ and μ being regular measures. $\square$

4 Chapter 5

4.1 Folland 5.1

Prove the following Proposition:

Proposition. 4.1:

If $\mathcal{X}$ is a normed vector space over K ($= \mathbb{R}$ or $\mathbb{C}$), then addition and scalar multiplication are continuous from $\mathcal{X} \times \mathcal{X}$ and $K \times \mathcal{X}$ to $\mathcal{X}$. Moreover, the norm is continuous from $\mathcal{X}$ to $[0, \infty)$; in fact, $\big|||x|| - ||y||\big| \leq ||x - y||$.

Proof. Let us define $A : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ as the addition map (I.e., defined as $A(x, y) = x + y$). The by construction, A is a linear map from the NVS $\mathcal{X} \times \mathcal{X}$ to the NVS $\mathcal{X}$. We thus will have (for $(x, y) \in \mathcal{X} \times \mathcal{X}$):

$$||A(x, y)|| = ||x + y|| \leq ||x|| + ||y|| \leq 2 \max\{||x||, ||y||\} = 2||(x, y)||$$

And therefore by (Folland) Proposition 5.2, since the above shows A is bounded, A must also be continuous.

Let now define $M : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$ as the scalar multiplication map (I.e., defined as $M(\alpha, x) = \alpha x$). Suppose now that $\epsilon > 0$ and we choose $\delta = \min\{1, \epsilon\}$. Then if $(\alpha, x) \in K \times \mathcal{X}$ such that $||(\alpha, x)|| < \delta$, we have:

$$\max\{|\alpha|, ||x||\} < \delta \leq \epsilon \Rightarrow ||M(\alpha, x)|| = ||\alpha x|| = |\alpha|||x|| < \delta^2 \leq \delta \leq \epsilon$$

And so is continuous at $(0, 0)$, and hence again by (Folland) Proposition 5.2, M is continuous.

Lastly, let again $\epsilon > 0$, but now set $\delta = \epsilon$. If $x, y \in \mathcal{X}$ such that $||x - y|| < \delta$, then:

$$||x|| = ||x - y + y|| \leq ||x - y|| + ||y|| \Rightarrow ||x|| - ||y|| \leq ||x - y||$$

And similarly for $||y|| - ||x|| \leq ||y - x|| = ||x - y||$. Therefore:

$$\big|||x|| - ||y||\big| \leq ||x - y|| < \delta = \epsilon$$

And so $||\cdot||$ is uniformly continuous and therefore continuous from $\mathcal{X}$ to $[0, \infty)$. $\square$

4.2 Folland 5.2

Prove the following Proposition:

Proposition. 4.2:

$L(\mathcal{X}, \mathcal{Y})$ is a vector space and the function $||\cdot||$ defined by (Folland, Equation 5.3) is a norm on it. In particular, the three expressions on the right of (5.3) are always equal.

Proof. We begin by defining:

$$||T||_1 := \sup\left\{||Tx|| \mid ||x|| = 1\right\}$$
$$||T||_2 := \sup\left\{\frac{||Tx||}{||x||} \mid x \neq 0\right\}$$
$$||T||_3 := \sup\left\{C \mid ||Tx|| \leq C||x|| \;\forall x \in \mathcal{X}\right\}$$

As in (Folland) Equation 5.3. We thus begin by showing $||\cdot||_1 = ||\cdot||_2 = ||\cdot||_3$. Firstly, if $x \in \mathcal{X}, x \neq 0$, then $\big|x/||x||\big| = 1$ and so $T(x)/||x|| = T(x/||x||) \leq ||\cdot||_1$. Since this is true $\forall x \in \mathcal{X}$, we may take the supremum and hence $||T||_2 \leq ||T||_1$. Next, again if $x \in \mathcal{X}$ and $||x|| = 1$, then $||Tx|| \leq ||T||_3$, and again taking the supremum implies $||T||_1 \leq ||T||_3$. Lastly, again supposing $x \in \mathcal{X}$, we simply have $||Tx|| \leq ||T||_2$, therefore $||T||_3 \leq ||T||_2$. Summarizing we have: $||T||_1 \leq ||T||_3 \leq ||T||_2 \leq ||T||_1$, and hence all our inequalities above are actually equalities, and proving the equivalence of the above forms of (5.3).

To prove that $||\cdot||$ does indeed define a norm, suppose $S, T \in L(\mathcal{X}, \mathcal{Y})$, and $x \in \mathcal{X}$. We thus have:

$$||(S+T)x|| = ||Sx + Tx|| \leq ||Sx|| + ||Tx|| \leq (||S|| + ||T||)||x|| \quad \Rightarrow \quad ||S+T|| \leq ||S|| + ||T||$$

If now $\alpha \in K(= \mathbb{C}$ or $\mathbb{R})$, then:

$$\begin{aligned}
&||\alpha Tx|| = |\alpha|||Tx|| \leq |\alpha|||T||||x|| \Rightarrow ||\alpha T|| \leq |\alpha|||T|| \\
\Rightarrow &||\alpha^{-1}(\alpha T)|| \leq |\alpha^{-1}|||\alpha T|| \Rightarrow |\alpha|||T|| \leq ||\alpha T|| \\
\Rightarrow &||\alpha T|| = |\alpha|||T||
\end{aligned}$$

And finally, $||T|| = 0 \iff ||Tx|| = 0 \;\forall x \in \mathcal{X}$ <u>and</u> $T \equiv 0$. Hence we've shown all the conditions for $||\cdot||$ to be a norm. $\square$

4.3 Folland 5.5

Prove the following Proposition:

Proposition. 4.3:

If $\mathcal{X}$ is a normed vector space, the closure of any subspace of $\mathcal{X}$ is a subspace.

Proof. Let X be a subspace of $\mathcal{X}$ and $\overline{X}$ denote its closure. Firstly, by definition, $0 \in \overline{\mathcal{X}}$. The other property that we need to show is that if that if $x, y \in \overline{X}$, and $a, b \in K$, then $ax + by \in \overline{X}$ as well. Since $x, y \in \overline{X}$, we know that $\exists \{x_j\}_1^\infty \subset X$ and $\{y_j\}_1^n \subset X$ s.t. $x_n \to x$ and $y_n \to y$ with respect to the norm, $||\cdot||$ on $\mathcal{X}$. So, $\forall \epsilon/2 > 0$, $\exists N_1, N_2 \in \mathbb{N}$ s.t. $||x_n - x|| < \epsilon/2$ and $||y_n - y|| < \epsilon/2 \;\forall n \geq N_1, N_2$ respectively. So, $\forall n \geq N = \max(N_1, N_2)$, we have:

$$\big|\big|(ax_n + by_n) - (ax + by)\big|\big| \leq ||ax_n - ax|| + ||by_n - by|| = |a|||x_n - x|| + |b|||y_n - y|| < 2\left(\frac{\epsilon}{2}\right) = \epsilon$$

And so since $ax_n + by_n \to ax + by$, and $ax_n + by_n \in X$ it implies $ax + by \in \overline{X}$ by the definition of $\overline{X}$. Therefore, $\overline{X}$ is indeed a subspace of $\mathcal{X}$. $\square$

4.4 Folland 5.6

Prove the following Proposition:

Proposition. 4.4:

Suppose that $\mathcal{X}$ is a finite-dimensional vector space. Let $e_1, \dots, e_n$ be a basis for $\mathcal{X}$ and define $||\sum_1^n a_j e_j||_1 = \sum_1^n |a_j|$.

a) $||\cdot||_1$ is a norm on $\mathcal{X}$.

b) The map $(a_1, \dots, a_n) \to \sum_1^n a_j e_j$ is a continuous form K^n with the usual Euclidean topology to $\mathcal{X}$ with the topology defined by $||\cdot||_1$.

c) $\{x \in \mathcal{X} \mid ||x||_1 = 1\}$ is compact in the topology defined by $||\cdot||_1$.

d) All norms on $\mathcal{X}$ are equivalent. (Compare any norm to $||\cdot||_1$.)

Proof.

a) We can first see that $||\mathbf{x}||_1 = 0 \iff \mathbf{x} = 0$ since $\sum_1^n |a_j| = 0 \iff a_j = 0\ \forall j = 1, \dots, n$, and $0 := 0e_1 + \cdots + 0e_n$.

Next, to see the triangle inequality, we first note that the triangle inequality naturally holds $\forall x, y \in K$. Therefore, if $\mathbf{x}, \mathbf{y} \in \mathcal{X} \Rightarrow \mathbf{x} = \sum_1^n \alpha_j e_j, \mathbf{y} = \sum_1^n \beta_j e_j$, and hence:

$$\begin{aligned} ||\mathbf{x}+\mathbf{y}||_1 &= \left|\left|\sum_{j=1}^n \alpha_j e_j + \sum_{j=1}^n \beta_j e_j\right|\right|_1 = \left|\left|\sum_{j=1}^n (\alpha_j + \beta_j) e_j\right|\right|_1 = \sum_{j=1}^n |\alpha_j + \beta_j| \\ &\leq \sum_{j=1}^n |\alpha_j| + \sum_{j=1}^n |\beta_j| = ||\mathbf{x}||_1 + ||\mathbf{y}||_1 \end{aligned}$$

So the triangle inequality holds. Now suppose $\lambda \in K$, we therefore have:

$$||\lambda \mathbf{x}||_1 = \left|\left|\lambda \sum_{j=1}^n \alpha_j e_j\right|\right|_1 = \left|\left|\sum_{j=1}^n (\lambda \alpha_j) e_j\right|\right|_1 = \sum_{j=1}^n |\lambda \alpha_j| = |\lambda| \sum_{j=1}^n |\alpha_j| = |\lambda| ||\mathbf{x}||_1$$

And hence we have shown the three conditions for $||\cdot||_1$ to be a norm on $\mathcal{X}$.

b) From Part a), by dropping the absolute values in expressions of the form $\sum_1^n |a_j|$, and replacing it by $\sum_1^n a_j e_j$, the one inequality now becomes an equality, and hence the rest proves that $T : K^n \to \mathcal{X}$, where $T(a_1, \cdots, a_n) = \sum_1^n a_j e_j$, is a <u>linear map</u>. We may now invoke (Folland) Proposition 5.2, which states T is continuous $\iff$ T is continuous at 0.

Let $\epsilon > 0$, and $\delta = \epsilon/n$. Then if:

$$||\mathbf{x} - 0|| = ||\mathbf{x}|| = (a_1^2 + \cdots a_n^2)^{1/2} < \delta \quad \Rightarrow \quad a_i^2 \leq (a_1^2 + \cdots + a_n^2) < \delta^2 \quad \forall i = 1, \dots, n$$

And so $|a_i| < |\delta| = \epsilon/n$. Therefore, we have:

$$||T\mathbf{x}||_1 = \left|\left|\sum_{j=1}^n a_j e_j\right|\right|_1 = |a_1| + \cdots + |a_n| < n\left(\frac{\epsilon}{n}\right) = \epsilon$$

c) We begin by showing $\Gamma := \{(a_1, \dots, a_n) \in K^n \mid \sum_1^n |a_j| = 1\} \subset K^n$ is compact. To see this, we can simply show that Γ is closed and bounded since $\Gamma \subset K^n = \mathbb{C}^n$ or $\mathbb{R}^n$. The boundness of Γ is easy to see since: $||(a_1, \dots, a_n)||_2 = ||x||_2 := (\sum_1^n a_j^2)^{1/2} \Rightarrow |a_j| \leq 1\ \forall j = 1, \dots, n \ \Rightarrow B_2(0) \supset \Gamma$, hence Γ is bounded.

To see Γ is closed, we show that Γ^c is open. If $\mathbf{x} \in \Gamma^c$, then:

$$\begin{aligned}
\mathbf{x} &\in \left\{(a_1, \ldots, a_n) \mid \sum_1^n |a_j| \neq 1\right\} \\
&\equiv \left\{(a_1, \ldots, a_n) \mid \sum_1^n |a_j| < 1\right\} \sqcup \left\{(a_1, \ldots, a_n) \mid \sum_1^n |a_j| > 1\right\} := \Gamma_1 \sqcup \Gamma_2
\end{aligned}$$

I.e., if $\mathbf{x} = (x_1, \ldots, x_n)$, then $\sum_1^n |x_j| < 1$ or $\sum_1^n |x_j| > 1$. Assume $\mathbf{x} \in \Gamma_1$, and $\mathbf{y} \in K^n$. Letting $\epsilon_1 = 1 - \sum_1^n |x_j| > 0$, then in taking $\delta_1 = \epsilon_1/n$, we have:

$$\begin{aligned}
||\mathbf{x} - \mathbf{y}||_2 < \delta_1 \quad &\Rightarrow \quad |x_i - y_i| \leq ||\mathbf{x} - \mathbf{y}||_2 < \delta_1 = \frac{\epsilon_1}{n} \quad \forall i \in \{1, \ldots, n\} \\
&\Rightarrow \quad ||\mathbf{x} - \mathbf{y}||_1 = \sum_{j=1}^n |x_j - y_j| < n\left(\frac{\epsilon}{n}\right) = 1 - \sum_{j=1}^n |x_j| \\
&\Rightarrow \sum_{j=1}^n |x_j| - \sum_{j=1}^n |y_j| < 1 - \sum_{j=1}^n |x_j| \qquad\qquad \text{since } |a| - |b| < |b - a| \\
&\Rightarrow \sum_{j=1}^n |y_j| < 1
\end{aligned}$$

And so $B_{\epsilon_1/n}(\mathbf{x}) \subset \Gamma_1$, so Γ_1 is open. Now suppose $\mathbf{x} \in \Gamma_2$. Letting $\epsilon_2 = \sum_1^n |x_j| - 1$ and $\mathbf{y} \in K^n$ as before. Then letting $\delta_2 = \epsilon_2/2$, we have:

$$\begin{aligned}
||\mathbf{x} - \mathbf{y}||_2 < \delta_2 \quad &\Rightarrow \quad |x_i - y_i| \leq ||\mathbf{x} - \mathbf{y}||_2 < \delta_2 = \frac{\epsilon_2}{n} \quad \forall i \in \{1, \ldots, n\} \\
&\Rightarrow \quad ||\mathbf{x} - \mathbf{y}||_1 = \sum_{j=1}^n |x_j - y_j| < n\left(\frac{\epsilon}{n}\right) = \sum_{j=1}^n |x_j| - 1 \\
&\Rightarrow \sum_{j=1}^n |x_j| - \sum_{j=1}^n |y_j| < \sum_{j=1}^n |x_j| - 1 \qquad\qquad \text{since } |b| - |a| < |b - a| \\
&\Rightarrow \sum_{j=1}^n |y_j| > 1
\end{aligned}$$

And so $B_{\epsilon_2/n}(\mathbf{x}) \subset \Gamma_2$, and hence Γ_2 is open. Now since $\Gamma^c = \Gamma_1 \sqcup \Gamma_2$, we can now conclude that Γ^c is open, and hence Γ is closed, and hence compact. Furthermore, in Part b), we showed that T (as defined in Part b) is continuous. Therefore, since:

$$T(\Gamma) = \{\mathbf{x} \in \mathcal{X} \mid ||\mathbf{x}||_1 = 1\}$$

We may now conclude that since Γ is compact, so too is $\{\mathbf{x} \in \mathcal{X} \mid ||\mathbf{x}||_1 = 1\}$ in the topology defined by $|| \cdot ||_1$.

d) Suppose $|| \cdot || : \chi \to \mathbb{R}_{\geq 0}$ is an arbitrary norm on $\mathcal{X}$. We recall that to show $|| \cdot ||$ and $|| \cdot ||_1$ are equivalent, we need to find $C_1, C_2 > 0$ s.t. $C_1||\mathbf{x}||_1 \leq ||\mathbf{x}|| \leq C_2||\mathbf{x}||_1$ $\forall \mathbf{x} \in \mathcal{X}$. If $\mathbf{x} = 0$, then $||\mathbf{x}||_1 = ||\mathbf{x}||$ since both are norms, selecting any $C_1 \leq C_2$ where $C_1, C_2 > 0$ proves the equivalence of these norms for $\mathbf{x} = 0$; therefore, assume $x \neq 0$.

If we let $C_2 = \max(\{||e_j||_1^n)$, then if $\mathbf{x} \in \mathcal{X} \ \Rightarrow \mathbf{x} = \sum_1^n x_j e_j$, then:

$$||\mathbf{x}|| \overset{*}{\leq} \sum_{j=1}^n |x_j|||e_j|| \leq C_2 \sum_{j=1}^n = C_2||\mathbf{x}||_1 \qquad \text{where we have } \overset{*}{\leq} \text{ from the } \Delta\text{-inequality}$$

So we have found an appropriate C_2.

We now claim that $||\cdot||$ is continuous in the topology defined by $||\cdot||_1$. To see this, let $\epsilon > 0$, and $\delta = \epsilon/n$. If $\mathbf{x}, \mathbf{y} \in \mathcal{X}$ and $||\mathbf{x} - \mathbf{y}||_1 < \delta$, then by what we found above:

$$||\mathbf{x} - \mathbf{y}|| \leq C_2||\mathbf{x} - \mathbf{y}||_1 < C_2\left(\frac{\epsilon}{C_2}\right) = \epsilon$$

Which tells us that $||\cdot||$ is indeed continuous on X in the topology defined by $||\cdot||_1$.

By Part c), we recall that $A := \{\mathbf{x} \in \mathcal{X} \mid ||\mathbf{x}||_1 = 1\}$ is a compact set in the topology defined by $||\cdot||_1$. Therefore, by the continuity of $||\cdot||$, and since we are assuming $x \neq 0$, we know that $\min_{x \in A} ||\mathbf{x}||$ exists, so let's call this min C_1. Explicitly now:

$$C_1 \leq \left|\left|\frac{\mathbf{x}}{||\mathbf{x}||_1}\right|\right| \quad \Rightarrow \quad C_1||\mathbf{x}||_1 \leq ||\mathbf{x}|| \quad \forall \mathbf{x} \in \mathcal{X}$$

Hence completing our proof since we found both C_1, C_2 which satisfy the necessary inequality.

□

4.5 Folland 5.9

Prove the following Proposition:

Proposition. 4.5:

Let $C^k([0,1])$ be the space of functions on $[0,1]$ possessing continuous derivatives up to order k on $[0,1]$, including one-sided derivatives at the endpoints.

a) If $f \in C([0,1])$, then $f \in C^k([0,1]) \iff f$ is k times continuously differentiable on $(0,1)$ and $\lim_{x \searrow n} f^{(j)}(x)$ and $\lim_{x \nearrow 1} f^{(j)}(x)$ exist for $j \leq k$. (The mean value theorem is useful.)

b) $||f|| = \sum_0^k ||f^{(j)}||_u$ is a norm on $C^k([0,1])$ that makes $C^k([0,1])$ into a Banach space. (Use induction on k. The essential point is that if $\{f_n\} \subset C^1([0,1])$, $f_n \to f$ uniformly, and $f_n' \to g$ uniformly, then $f \in C^1([0,1])$ and $f' = g$. The easy way to prove this is to show that $f(x) - f(0) = \int_0^x g(t)dt$.)

Proof.

a) We'll proceed to prove this claim through induction. Suppose $k = 0$, then the forward case of $f \in C([0,1])$ implying f is differentiable on $(0,1)$ and and $\lim_{x \searrow n} f(x)$ and $\lim_{x \nearrow 1} f(x)$ existing is by the definition of $C([0,1])$.

Now, for the backward direction ($k = 0$), suppose $f \in C((0,1))$, $\lim_{x \searrow n} f(x)$, and $\lim_{x \nearrow 1} f(x)$ exist - this, however, is simply the definition of $f \in C([0,1])$.

Let $L_0^{(j)} := \lim_{x \searrow n}^{(j)} f(x)$ and $L_1^{(j)} := \lim_{x \nearrow n} f^{(j)}(x)$. Now assume the property above holds for $k = n-1$. The forward direction is simply by definition. For the backward direction, if we wish to show that f being k times differentiable on $(0,1)$ and $\lim_{x \searrow n}^{(j)} f(x)$ and $\lim_{x \nearrow n} f^{(j)}(x)$ existing for $j \leq n$ implies $f \in C^k([0,1])$, we may proceed as follows. Firstly, by the existence of the one sided derivatives, we know that $\forall \epsilon > 0$, $\exists \delta > 0$ such that if $0 < x < \delta$, then $|f^{(j)}(x) - L_0^{(j)}| < \epsilon, \forall j \leq n$. Furthermore, WLOG, we may omit the $L_1^{(j)}$ case since all we need to chance in the argument is that $\delta < x < 1$ instead of $0 < x < \delta$. Moreover, by the mean value theorem, $\exists \hat{x} \in (0,\delta)$ s.t. $f^{(j-1)}(x) - f^{(j-1)}(0) = (x-0)f^{(j)}(\hat{x}) = xf^{(j)}(\hat{x})$ Therefore:

$$\left| \frac{f^{(j-1)}(x) - f^{(j-1)}(x)}{x} - L_0 \right| = |f^{(j)}(\hat{x}) - L_1| < \epsilon \quad \text{since } \hat{x} \in (0,\delta)$$

And so $\lim_{x\searrow 0} \frac{f^{(j-1)}(x)-f^{(j-1)}(x)}{x} = L_0 \ \forall j \leq n$, and by the exact same argument for 1, we see that $\lim_{x\nearrow 0} \frac{f^{(j-1)}(x)-f^{(j-1)}(x)}{x} = L_1 \ \forall j \leq n$, and so $f \in C^{(n)}([0,1])$, completing our inductive step and proving this proposition $\forall k \in \mathbb{N}$.

b) We proceed, as hinted, by induction. Suppose that $\{f_n\}_1^\infty$ is Cauchy in C^1. Then $f_n \to f$ in C^0 and $f_n' \to g$ in C. Therefore:

$$f_n(x) = f_n(0) + \int_0^x f_n'(y)dy$$

However, since $f_n' \to g$, by the dominated convergence theorem we have:

$$f(x) = \lim_{n\to\infty} f(x) = \lim_{n\to\infty} f_n(0) + \lim_{n\to\infty} \int_0^x f_n'(y)dy = f(0) + \int_0^x g(y)dy$$

And therefore by the fundamental theorem of calculus we may conclude that $g = f'$, and so $f_n \to f$ in C^1.

We now make our inductive step. Assume the statement is true up until $j = k$. Suppose then that $\{f_n\}_1^\infty$ is Cauchy in C^{k+1} and $f_n \to f$ in C^k and $f_n^{(k+1)} \to g$ in C^{k+1}. Therefore, $f_n^{(k)} \to f_n^{(k+1)}$ in C, and $f_n^{(k+1)} \to g$ in C. We therefore may conclude that $f_n^{(k+1)} \to f_n^{(k+1)}$ in C, and ultimately $f_n \to f$ in $C^{(k+1)}$.

To finish our proof, we need to prove that $||\cdot||$ is indeed a norm. Firstly, if $f \not\equiv 0$, then naturally $||f|| \neq 0$, so $||f|| = 0 \iff f \equiv 0$. Since $||\cdot||$ is simply a sum of other norms, the triangle inequality and absolutely scalability are both trivially immediate like definiteness.

□

5 Chapter 6

5.1 Folland 6.3

Prove the following Proposition:

Proposition. 5.1:

If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm $||f|| = ||f||_p + ||f||_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Proof. We begin by first showing that $L^p \cap L^r$ is a Banach Space w.r.t. $||f|| = ||f||_p + ||f||_r$ (I.e., show $L^p \cap L^r$ a normed vector space and complete w.r.t. $||f||$).

The fact that $||\cdot||_r$ and $||\cdot||_p$ are norms implies $||\cdot||$ is a norm. Firstly, $||\cdot|| \geq 0$ since $||\cdot||_p, ||\cdot||_r \geq 0$. Now, suppose $f, g \in L^r \cap L^p$, and $\lambda \in K$, then we have:

$$||f+g|| = ||f+g||_p + ||f+g||_r \leq ||f||_p + ||g||_p + ||f||_r + ||g||_r = ||f|| + ||g||$$

$$||\lambda f|| = ||\lambda f||_p + ||\lambda f||_r = |\lambda|\ ||f||_p + |\lambda|\ ||f||_r = |\lambda|\ ||f||$$

$$||f|| = 0 \iff ||f||_p = ||f||_r = 0 \iff f \equiv 0\ \mu\text{-a.e.}$$

We can also immediately see that $L^p \cap L^r$ is a vector space since if $u, v \in L^p \cap L^r$, then $u, v \in L^p$ and L^r, and so all our conditions for being a vector subspace are satisfied since both L^p and L^r are vector subspaces.

Suppose now that $\{f_n\}_1^\infty$ be a Cauchy sequence in $L^p \cap L^r$. By noting that $\forall n, m \in \mathbb{N}$, we have $||f_n - f_m||_p \le ||f_n - f_m||$ and $||f_n - f_m||_r \le ||f_n - f_m||$, and hence $\{f_n\}_1^\infty$ are also Cauchy in L^p and L^r. We can thus define g and h as $\lim f_n$ in L^p and L^r respectively. Let $\epsilon > 0$, then $\exists\ N \in \mathbb{N}$ s.t. if we take $\delta = \epsilon^{(p+1)/p}$, then letting $||f_n - g||_p < \delta$, and in setting $E := \{x \in \mathfrak{X} \mid \epsilon \le |f_n(x) - g(x)|\}$, we have:

$$\mu(E) = \frac{1}{\epsilon^p}\int_E \epsilon^p d\mu \le \frac{1}{\epsilon^p}\int_E |f_n - g|^p d\mu \le \frac{1}{\epsilon^p}\int |f_n - g|^p d\mu = \frac{1}{\epsilon^p}\big(||f_n - g||_p\big)^p < \frac{1}{\epsilon^p}\big(\delta\big)^p = \epsilon$$

I.e., $\mu(E) < \epsilon \Rightarrow \{f_n\}_1^\infty$ converges in measure to g. If $r < \infty$, the argument holds for interchanging p for r. If $r = \infty$, then $\exists$ a subsequence f_{n_k} of $\{f_n\}_1^\infty$ s.t. $f_{n_k} \to h$ μ-a.e. We have therefore shown that $g = h$, and so $g \in L^p \cap L^r$. Therefore, since $f_n \to g$ in L^p and L^r, we have $f_n \to g$ in $L^p \cap L^r$ - and hence $L^p \cap L^r$ is a Banach space with norm $||\cdot||$.

Let now $p < q < r$. By (Folland) Proposition 6.10, we know that $\exists \lambda \in (0,1)$ s.t.$||f||_p^\lambda ||f||_r^{1-\lambda}$ where $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. Thus, since $||f||_p \le ||f||$ and $||f||_r \le ||f||$, we have:

$$||f||_q \le ||f||_p^\lambda ||f||_r^{1-\lambda} \le ||f||^\lambda ||f||^{1-\lambda} = ||f||$$

Suppose now that $\epsilon > 0$ and $f, g \in L^p \cap L^r$, then if $||f - g|| < \delta = \epsilon$, we have $||f - g||_q \le ||f - g|| < \epsilon$ by the above inequality. Hence $\iota : L^p \cap L^r \to L^q$ is uniformly continuous (and naturally continuous as well). $\square$

5.2 Folland 6.4

Prove the following Proposition:

Proposition. 5.2:

If $1 \le p < r \le \infty$, $L^p + L^r$ is a Banach Space with norm $||f|| = \inf\{||g||_p + ||h||_r \mid f = g + h\}$, and if $p < q < r$, the inclusion map $L^q \to L^p + L^r$ is continuous.

Proof. We begin by showing $||\cdot||$, as defined, is a norm. Firstly, $||\cdot|| \ge 0$ since $||\cdot||_p, ||\cdot||_r \ge 0$. Now, suppose $f_1, f_2 \in L^r + L^p$, and $\lambda \in K$, then we have:

$$\begin{aligned}
||f_1 + f_2|| &= \inf\Big\{||g||_p + ||h||_r \ \Big|\ f_1 + f_2 = g + h\Big\} \\
&= \inf\Big\{||g_1 + g_2||_p + ||h_1 + h_2||_r \ \Big|\ f_1 + f_2 = g + h = (g_1 + g_2) + (h_1 + h_2)\Big\} \\
&\le \inf\Big\{\big(||g_1||_p + ||g_2||_p\big) + \big(||h_1||_r + ||h_2||_r\big) \ \Big|\ f_1 + f_2 = g + h = (g_1 + g_2) + (h_1 + h_2)\Big\} \\
&\le \inf\Big\{||g_1||_p + ||h_1||_r \Big|\ f_1 = g_1 + h_1\Big\} + \inf\Big\{||g_2||_p + ||h_2||_r \Big|\ f_2 = g_2 + h_2\Big\} \\
&= ||f_1|| + ||f_2||
\end{aligned}$$

$$\begin{aligned}||\lambda f|| &= \inf\left\{||\lambda g||_p + ||\lambda h||_r \;\middle|\; \lambda f = \lambda(g+h)\right\} \\ &= \inf\left\{|\lambda|\;||g||_p + |\lambda|\;||h||_r \;\middle|\; \lambda f = \lambda(g+h)\right\} \\ &= |\lambda| \inf\left\{||g||_p + ||h||_r \;\middle|\; f = g+h\right\} \\ &= |\lambda|\;||f||\end{aligned}$$

$$||f|| = 0 \iff ||f||_p = ||f||_r = 0 \;\forall g,h \text{ s.t. } f = g+h \iff f \equiv 0 \;\mu\text{-a.e.}$$

We can also immediately see that $L^p + L^r$ is a vector space since if $u, v \in L^p + L^r$, then $u = u_1 + u_2, v = v_1 + v_2$ where $u_1, v_1 \in L^p$ and $u_2, v_2 \in L^r$, and so all our conditions for being a vector subspace are satisfied since both L^p and L^r are vector subspaces.

To show completeness, we make use of (Folland) Theorem 5.1 which states that a normed vector space, $\mathfrak{X}$, is complete $\iff$ every absolutely convergent series in $\mathfrak{X}$ converges. So, suppose $\sum_1^\infty f_n$ be an absolutely convergent series in $L^p + L^r$. By the definition of inf and $||\cdot||$, we know that $\forall n \in \mathbb{N}$, $\exists\, g_n \in L^p, h_n \in L^r$ s.t. $f_n = g_n + h_n$ where $||g_n||_p + ||h_n||_r < ||f_n|| + 2^{-n}$. Therefore, from this inequality, and since both $\sum_1^\infty f_n$ and $\sum_1^\infty 2^{-n}$ are absolutely convergent, so too will $\sum_1^\infty g_n$ and $\sum_1^\infty h_n$. Since L^p and L^r are Banach spaces, $\sum_1^N g_n \to g \in L^p$ and $\sum_1^N h_n \to h \in L^r$. Furthermore, by definition $||\cdot|| \le ||\cdot||_p$ and $||\cdot|| \le ||\cdot||_r$, so combining these two reverse inequalities, we have $\sum_1^\infty f_n = \sum_1^\infty (g_n + h_n)$, which therefore has a limit in $L^p + L^r$, explicitly $g + h \in L^p + L^r$. We have thus show all the necessary conditions for $L^p + L^r$ to be a Banach Space w.r.t. $||\cdot||$.

Suppose $p < q < r$ and $f \in L^q$. Let $E := \{x \in \mathfrak{X} \mid 1 < |f(x)|\}$. Thus, by the construction of E, we therefore have: $|f\chi_E|^p \le |f\chi_E|^q$ and $|f\chi_{E^c}|^p \le |f\chi_{E^c}|^q$ (I.e., $f\chi_E \in L^p, f\chi_E \in L^r$), and hence:

$$||f|| = ||f\chi_E + f\chi_{E^c}|| \le ||f\chi_E||_p + ||f\chi_E||_r \le ||f\chi_E||_q + ||f\chi_{E^c}||_q = ||f||_q$$

Suppose now that $\epsilon > 0$ and $f, g \in L^q$, then if $||f-g||_q < \delta = \epsilon$, we have $||f-g|| \le ||f-g||_q < \epsilon$ by the above inequality. Hence $\iota : L^q \to L^p + L^r$ is uniformly continuous (and naturally continuous as well).

□

5.3 Folland 6.5

Prove the following Proposition:

Proposition. 5.3:

Suppose $0 < p < q < \infty$. Then:

a) $L^p \not\subset L^q \iff X$ contains sets of arbitrarily small positive measure.

b) $L^q \not\subset L^p \iff X$ contains sets of arbitrarily large finite measure.

c) What about the case of $q = \infty$?

Proof.

a) We first prove the following Lemma:

Lemma. 5.1: Chebyshev's Inequality

$$\mu(E_t) \le \left(\frac{||f||_p}{t}\right)^p$$

Where $E_t = \{x \in X \mid |f(x)| \ge t\}$ and $p \in (0, \infty)$.

Proof. Let $g(x) = x^p$ if $x \geq t$, and 0 otherwise. We thus have $0 \leq t^p \chi_{E_t} \leq |f|^p \chi_{E_t}$, and hence:

$$\mu(E_t) = \frac{1}{t^p} \int t^p \chi_{E_t} d\mu \leq \frac{1}{t^p} \int_{E_t} |f|^p d\mu \leq \frac{\left(||f||_p\right)^p}{t^p}$$

□

Now back to the problem at hand. For the forward direction, we proceed via the contrapositive, I.e., suppose $\exists \epsilon > 0$ s.t. $\forall E \subset \mathcal{M}(X), \mu(E) \notin (0, \epsilon)$. From Chebyshev's Inequality, we know that $\exists T$ s.t. $\forall t \geq T$, $\mu(E_t) = 0$ since $\mu(E_t) \leq \left(\frac{||f||_p}{t}\right)^p \to 0$, and so $|f| \leq T$ a.e. So:

$$\int |f|^q d\mu = \int_{E_t} |f|^q d\mu + \int_{E_t^c} |f|^q d\mu \leq T^q \mu(E_t) + \int_{E_t^c} |f|^p d\mu < \infty$$

And so $f \in L^q$.

For the converse, suppose $\forall \epsilon > 0$, $\exists E \in \mathcal{M}(X)$ s.t. $\mu(E) \in (0, \epsilon)$. Let us define $\{F_n\}_1^\infty$ where $0 < \mu(F_n) < 1/n$ so that $\mu(F_n) \to 0$. By defining $G_n := F_n \setminus \cup_{n+1}^\infty F_m$, we must have $0 < \mu(F_n) \leq \mu(\cup_n^\infty G_m)$. Furthermore, by taking subsequences, we may actually assume now that $0 < \mu(G_m) \leq 2^{-m}$. Now if we define:

$$f := \sum_{n=1}^{\infty} \left(\mu(G_n)\right)^{-1/q} \chi_{G_n} n^{-2/p} \quad (\geq 0)$$

Then we have:

$$\int |f|^p d\mu = \int f^p d\mu = \int \sum_{n=1}^{\infty} \left(\mu(G_n)\right)^{-p/q} \chi_{G_n} n^{-2} d\mu = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} < \infty$$

And so $f \in L^p$; however, one can see that $f \notin L^q$ since:

$$\int |f|^q d\mu = \int f^q d\mu = \sum_{n=1}^{\infty} \left(\mu(G_n)\right)^{1-p/q} n^{-2q/p} \geq \sum_{n=1}^{\infty} 2^{p/q-1} n^{-2q/p} = \infty$$

b) For the forward direction, the proof here is completely analogous to that in a). For the converse, by substituting $\left(\mu(G_n)\right)^{-1/q}$ for $\left(\mu(G_n)\right)^{-1/(p+1)}$, and noting that now we have $2^m \leq \mu(G_m) < \infty$ instead of $0 < \mu(G_m) \leq 2^{-m}$, the same results as in a) still hold.

c) For the case of $q = \infty$, we have $L^\infty \not\subset L^p \iff \mu(X) = \infty$, since if $|f|^p < C \in \mathbb{R}_{\geq 0}$, we have:

$$\int |f|^p d\mu \leq C \int d\mu \leq C\mu(X) < \infty$$

□

5.4 Folland 6.7

Prove the following Proposition:

Proposition. 5.4:

If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ $\forall q > p$, then $||f||_\infty = \lim_{q \to \infty} ||f||_q$.

Proof. We may first assume $f \not\equiv 0$ a.e. by the triviality of this case. From (the proof of Folland) Proposition 6.10, we know that:

$$||f||_q \leq \big(||f||_\infty\big)^{1-p/q}\big(||f||_p\big)^{p/q}$$

And so:

$$\limsup_{q\to\infty} ||f||_q \leq \limsup_{q\to\infty} \Big(\big(||f||_\infty\big)^{1-p/q}\big(||f||_p\big)^{p/q}\Big) = ||f||_\infty$$

Furthermore, by our initial assumption, we have $||f||_\infty > 0$. Suppose now that $0 < a < ||f||_\infty$ and $E_a := \{x \in \mathfrak{X} \mid |f(x)| \geq a\}$. We thus have:

$$\begin{aligned} a^q\mu(E_a) \leq \big(||f||_p\big)^p \leq \int_{E_a} |f|^q d\mu \leq \big(||f||_q\big)^q \quad &\Rightarrow \quad \big(a^q\mu(E_a)\big)^{1/q} \leq \Big(\big(||f||_q\big)^q\Big)^{1/q} \\ &\Rightarrow \quad \liminf_{q\to\infty} a\big(\mu(E_a)\big)^{1/q} \leq \liminf_{q\to\infty} ||f||_q \\ &\Rightarrow \quad a \leq \liminf_{q\to\infty} ||f||_q \end{aligned}$$

And so letting $a \to ||f||_\infty$, we thus have:

$$\limsup_{q\to\infty} ||f||_q \leq ||f||_\infty \leq \liminf_{q\to\infty} ||f||_q$$

And so we must have all our inequalities become equalities: hence $\lim_{q\to\infty} ||f||_q = ||f||_\infty$.

□

5.5 Folland 6.10

Prove the following Proposition:

Proposition. 5.5:

Suppose $1 \leq p < \infty$. If $f_n, f \in L^p$ and $f_n \to f$ a.e., then $||f_n - f||_p \to 0 \iff ||f_n||_p \to ||f||_p$. [Use Exercise 20 in (Folland) 2.3.]

Proof. For the forward direction, if $||f_n - f||_p \to 0$, by the triangle inequality we have:

$$\big|||f_n||_p - ||f||_p\big| \leq ||f_n - f||_p \to 0$$

And we therefore have $||f_n||_p \to ||f||_p$.

For the converse, suppose $||f_n||_p \to ||f||_p$. We now quickly prove the following result:

$$\text{If } z, w \in \mathbb{C}, \text{ then } |z-w|^p \leq 2^{p-1}(|z|^p + |w|^p)\ \forall p \geq 1$$

By the second derivative test, $g(z) = |z|^p$ is convex (I.e., $g(tz + (1-t)w) \leq tg(z) + (1-t)g(w)$). So, if we set $t = 1/2$, and move the 2^p over to the other side, we have:

$$|z-w|^p \leq 2^{p-1}(|z|^p + |w|^p) \iff \left|\frac{z-w}{2}\right|^p \leq \frac{1}{2}|z|^p + \frac{1}{2}|w|^p$$

For which the latter is recognizably true due to the convexity of $|\cdot|^p$ for $p \geq 1$ (and in making a change of variables $w' = -w$)

Carrying on, let us define $g_n := 2^{p-1}(|f|^p + |f_n|^p) - |f - f_n|^p$. By the above inequality, we know that $g_n \geq 0$, and so we may apply Fatou's Lemma:

$$2^p\big(||f||_p\big)^p \leq \liminf_{n\to\infty} \int g_n = 2^p\big(||f||_p\big)^p - \limsup_{n\to\infty} \int |f - f_n|^p d\mu$$

And so $\limsup \int |f - f_n|^p d\mu \leq 0 \Rightarrow ||f - f_n||_p \to 0$.

□

5.6 Folland 6.14

Prove the following Proposition:

Proposition. 5.6:

If $g \in L^\infty$, the operator T defined by $Tf = fg$ is bounded on L^p for $1 \le p \le \infty$. Its operator norm is at most $||g||_\infty$ with equality if μ is semi-finite.

Proof. Firstly, we may assume $g \not\equiv 0$ due to the triviality of this case. We now proceed to see that:

$$\big(||Tf||_p\big)^p = \int |fg|^p d\mu = |f|^p |g|^p d\mu \le \big(||g||_\infty\big)^p \int |f|^p d\mu = \big(||g|_\infty|\big)^p \big(||f||_p\big)^p \quad [\le \text{ since } |g| \le ||g||_\infty]$$

$$\Rightarrow ||T|| \le ||g||_\infty$$

To see equality if μ is semi-finite, suppose $0 < \epsilon < ||g||_\infty$, By μ's semi-finitness, $\exists$ E s.t. $||g||_\infty - \epsilon < |g|$ $\forall x \in E$. Thus, we have:

$$||T\chi_E||_p = ||g\chi_E|| > \big(||g||_\infty - \epsilon\big)||\chi_E||_p \quad \Rightarrow \quad ||T|| > ||g||_\infty - \epsilon \quad \Rightarrow \quad ||T|| \ge ||g||_\infty$$

Where we have the last implication by ϵ's arbitrarily, and to satisfy both equalities, we must have $||g||_\infty = ||T||$. $\square$