Sample Problems for MIDTERM EXAM I,
Ma 553

1. Does the symmetric group S_5 have a subgroup of order 10? Justify your answer.

2. Describe the Sylow p-subgroups in A_4, S_4, A_5,S_5 and find their number.

3. Let G be a subgroup generated by 5-sycle in S_5. Find the order of $N_{S_5}(G)$.

4. (a) List representatives for each conjugacy class in the symmetric group S_4 and state the number of elements in each conjugacy class.

   (b) List representatives for each conjugacy class in the alternating group A_4 and state the number of elements in each conjugacy class.

   (c) Determine the number of elements of order 2 in the symmetric group S_5.

   (d) Determine the number of elements of order 2 in the symmetric group A_5.

   (e) Find the number of 2-subgroups in A_4.

5. Show that for any element σ of order 2 in the alternating group A_n. there exists $\tau \in S_n$ such that $\tau^2 = \sigma$.

6. Let G be a group having order $2k$, where k is an odd integer. Prove that G has a subgroup of order k.

7. (a) What is the order of the group $GL_3(F_3)$ of 3×3 invertible matrices with entries in F_3 ?

   (b) What is the order of the group $GL_3(F_3)$ of 3×3 invertible upper triangular matrices with entries in F_3 ?

   (c) Does there exist a nonabelian group of order 27 ? Justify your answer.

8. Prove that the order of the automorphism group of $(Z/3)^4$ is $80 \times 78 \times 72 \times 54$. Describe an example of a group of order 405 which is not the direct product of Z_5 with a group of order 81. Justify your answer.

9. Let G be a group of order p^3 ,where p is prime, and suppose H is a subgroup of G with $|H| = p$. Prove or disprove that there must exist a subgroup K of G such that $H \subseteq K$ and $|K| = p^2$.

10. Let G be a finite group, $p > 0$ a prime number, and H a normal subgroup of G. Prove the following assertions.

    (a) Any Sylow p-subgroup of H is the intersection $P \cap H$ of a Sylow p-subgroup P of G and subgroup H.

    (b) Any Sylow p-subgroup of G/H is the is the quotient PH/H , where P is a Sylow p-subgroup of G.

11. Let H be a normal subgroup of a finite group G, and let $N \subset H$ be a normal Sylow subgroup of H. Prove that N is a normal subgroup of G.

12. Prove that there is no simple group of order $5^3 \cdot k$ which has a subgroup of index 8.

13. Let G be a group of order 105.

    (a) Show that G has a normal subgroup of order 5 or 7.

    (b) Show that G has a cyclic normal subgroup of order 35.

    (c) Show that the Sylow 5- and 7-subgroups of G are both normal.

    (d) Classify groups of order 105.

14. Let G be a group of order 66.

    (a) Show that G has a unique subgroup of order 11.

    (b) Show that G has a cyclic subgroup-call it H-of order 33.

    (c) Show that G has a unique subgroup of order 3.

    (d) How many elements of order ≤ 2 does the automorphism group of H have? (Justify your answer.)

(e) Any group of order 66 is isomorphic to one and only one of Z_{66}, $S_3 \times Z_{11}$, $Z_3 \times D_{22}$, or D_{66},

15. Show that a simple group which has a subgroup of index $n > 2$ is isomorphic to a subgroup of the alternating group A_n.

16. Let G be a finite group, let p be a prime divisor of the order $|G|$, and let P be a Sylow p-subgroup of G. Let $N_G(P)$ be the normalizer of P, and $C_G(P) \subset N_G(P)$ the centralizer of P.

    (a) Show that the index $[N_G(P) : C_G(P)]$ is the order of a subgroup of the automorphism group of P.

    (b) Show that p divides $[N_G(P) : C_G(P)]$ if, and only if, P is non-abelian. (Hint: P is abelian iff $C_G(P) \supset P$.)

    (c) Show that if P is cyclic and the $\gcd(|G|, p-1) = 1$ then $C_G(P) = N_G(P)$. (Hint: Consider conjugation morphism $N_G(P) \to \mathrm{Aut}(P)$)

17. Show that any group G of order 80 is solvable.

18. Assume G contains a normal Sylow 2-subgroup P which is a cyclic group, and such that G/P is cyclic.

    (a) Show that the action of G by conjugation on P its trivial.(10 points) (Hint: consider the induced action of G/P on P.)

    (b) Show that G is abelian.

19. Show that P is abelian whenever $\mathrm{Aut}(P)$ is cyclic. (Hint: A subgroup of cyclic group is cyclic)

20. (1) Find all (up to isomorphism) abelian groups of order 40 (10 points).
    (2) Find the number of elements of order 2 in each of them (10 points).