A NOTE FOR REAL ANALYSIS QUALIFYING EXAM IN TAMU

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Abstract. This note contains solutions to the questions occurred in past Real analysis qualifying exams from Jan 2009 to Jan 2017. I did most of them. The rest are folklore. Typos and errors are inevitable. Comments and corrections are welcome.

1. January 2017

Problem 1.1. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Show that if for all n, $\mu\{x : |f_n(x)| > 1/n\} < n^{-3/2}$, then $f_n \to 0$ a.e. (μ).

Proof. Define $M = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x : |f_n(x)| > 1/n\}$. Then $\mu(\bigcup_{n=m}^{\infty} \{x : |f_n(x)| > 1/n\}) \leq \sum_{n=m}^{\infty} n^{-3/2} \to 0$ as $m \to \infty$. Then $\mu(M) = 0$. Consider $x \in M^c$ iff there is a m such that for all $n > m$, $|f_n(x)| \leq 1/n$, i.e. $f_n \to 0$ a.e. □

Problem 1.2. Find all $f \in L^1(1,2)$ such that for all $n \in \mathbb{N}$, $\int_1^2 x^{2n} f = 0$.

Proof. We firstly extend f to be defined on $[1,2]$ by setting $f(1) = 0$ and $f(2) = 0$. We still denote this function f. Now, $f \in L^1[1,2]$ and we still have $\int_1^2 x^{2n} f = 0$. Then, by Stone-Weirtrass theorem, we have for any continuous function $g \in C[1,2]$, $\int_1^2 gf = 0$. Then use the same argument occurred in problem 2 in Jan 2016. We can see $f = 0$ a.e. (μ). □

Problem 1.3.

Problem 1.4. We say a sequence $\{a_n\}$ in $[0,1]$ is equi-distributed if for all interval $[c,d] \subset [0,1]$, $\lim_{n\to\infty} \frac{|\{a_1,\ldots,a_n\} \bigcap [c,d]|}{n} = d - c$. Prove that $\{a_n\}$ in $[0,1]$ is equi-distributed iff $\lim_{n\to\infty} \int f d\mu_n = \int f dm$ for all $f \in C[0,1]$, where $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{a_n}$

Proof. By the definition, we see that $\{a_n\}$ is equi-distributed iff for all interval $[c,d] \subset [0,1]$, $\lim_{n\to\infty} \int \chi_{[c,d]} d\mu_n = \int \chi_{[c,d]} dm$.

Then if $\lim_{n\to\infty} \int f d\mu_n = \int f dm$ for all $f \in C[0,1]$. For any interval $[c,d]$, define continuous functions $1 \geq f_n \geq 0$ ($n > N$ for some proper N) with $f_n = 1$ on $[c,d]$ and $supp(f_n) \subset [c-1/n, d+1/n]$ such that $f_n \downarrow \chi_{[c,d]}$. Then for all $\epsilon > 0$, there is a K such that whenever $k > K$, $\mu_k[c,d] \leq \int f_n d\mu_k \leq \int f_n d\mu + \epsilon \leq \mu[c,d] + 2/n + \epsilon$.

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Note that this K depend on the interval $[c,d]$. Use this argument for $[c-1/n,c]$ and $[d,d+1/n]$, for the ϵ, for each n, there is a K_n such that whenever $k > K_n$, $\int|f_n-\chi_{[c,d]}|d\mu_k \leq \int\chi_{[c-1/n,c]}d\mu_k+\int\chi_{[d,d+1/n]}d\mu_k \leq 4/n+\epsilon$. In addition, $\int|f_n-\chi_{[c,d]}|dm \leq 2/n$ also holds. Now, fix a n big enough such that $6/n<\epsilon$ and $|\int f_n d\mu_k - \int f_n dm| < \epsilon$. This implies that whenever $k > K_n$, $|\int\chi_{[c,d]}d\mu_k - \int\chi_{[c,d]}dm| \leq \int|f_n-\chi_{[c,d]}|d\mu_k+|\int f_n d\mu_k-\int f_n dm|+\int|f_n-\chi_{[c,d]}|dm \leq 6/n+2\epsilon < 3\epsilon$. We are done.

In the converse, If for all interval $[c,d]\subset[0,1]$, $\lim_{n\to\infty}\int\chi_{[c,d]}d\mu_n = \int\chi_{[c,d]}dm$. Then this pass to all step functions. For any continuous function $f\in C[0,1]$, we can use step functions to approximate f under the norm $\|\cdot\|_\infty$. say for every $\epsilon>0$, there is a $0=x_0<x_1<...<x_N=1$ such that for all $n\leq N$, $|\max_{x_n\leq x\leq x_{n+1}} f(x) - \min_{x_n\leq x\leq x_{n+1}} f(x)| < \epsilon$. Now, define $g=\sum_{n=0}^N a_n\chi_{[x_n,x_{n+1}]}$ where a_n is an arbitrary number between $\min_{x_n\leq x\leq x_{n+1}} f(x)$ and $\max_{x_n\leq x\leq x_{n+1}} f(x)$. Thus $\|g-f\|_\infty<\epsilon$. Now, for the ϵ, there is a K such that whenever $k>K$, $|\int g d\mu_k - \int g dm| < \epsilon$ and thus $|\int f d\mu_k - \int f dm| \leq \int|f d\mu_k - \int g d\mu_k| + |\int g d\mu_k - \int g dm| + |f dm - \int g dm| < 2\|f-g\|_\infty+\epsilon < 3\epsilon$. We are done. □

Problem 1.5. Let S be a closed subspace of $(C[0,1],\|\cdot\|_\infty)$. If S is also closed under $\|\cdot\|_\infty$, then show S is finite-dimensional. (This is question 66 in Folland on page 178, see also mathoverflow 52509 for other solutions.)

Proof. Consider the identity map $id : (S,\|\cdot\|_\infty)\to(S,\|\cdot\|_2)$ is bounded since $\|f\|_2 \leq \|f\|_\infty$ in general (see problem 8 in Jan 2016). Then by open mapping theorem. $\|\cdot\|_2$ is equivalent to $\|\cdot\|_\infty$ on S, say for all $f\in S$, $\|f\|_\infty \leq C\|f\|_2$. Note that S is a Hilbert space. Let $\{f_n\}_{n\in I}$ be an orthonormal basis of S. For all $f\in S$, $x\in[0,1]$. The evaluation map $\delta_x(f)=f(x)$ is bounded and linear w.r.t $\|\cdot\|_2$. Indeed, $|\delta_x(f)|=|f(x)|\leq\|f\|_\infty\leq C\|f\|_2$. Then, by Riesz's lemma, there is a function $g_x\in C[0,1]$ such that $f(x)=\langle f,g_x\rangle$ with $\|g_x\|\leq C$

Thus, $\sum_{n\in I}|f(x)|^2=\sum_{n\in I}|\langle f,g_x\rangle|^2=\|g_x\|_2^2\leq C^2$. Then integration implies that $|I|\leq C^2$. □

Problem 1.6.

Problem 1.7.

Problem 1.8. (1) Construct a Lebesgue measurable set $A\subset\mathbb{R}$ so that for all $a<b$ $0<m(A\bigcap[a,b])<b-a$.
(2) Suppose that a Lebesgue measurable set $A\subset\mathbb{R}$ so that for all $a<b$, $m(A\bigcap[a,b]) < (b-a)/2$. Prove that $m(A)=0$.

Proof. (1) It suffices to show the case $A \subset [0,1]$ since then $B = \bigcup_{n\in\mathbb{Z}} n + A$ works. It also suffices to consider all intervals with rational endpoints by density of $\mathbb{Q}$. Now fix an enumeration of all this subintervals of $[0,1]$, say $\{I_n : n \in \mathbb{N}\}$. We define sequences of generalized cantor set $\{C_n\}$ and $\{D_n\}$ such that (i) $m(C_n) > 0$ and $m(D_n) > 0$; (ii) $C_n \bigcup D_n \subset I_n$; (iii) $(\bigcup_{k=1}^{n} C_k) \bigcap (\bigcup_{k=1}^{n} D_k) = \emptyset$.

We define these sequence by induction. If $C_1, ..., C_n$ and $D_1, ..., D_n$ are defined. They are nowhere dense closed sets. Then $\bigcup_{k=1}^{n} C_k \bigcup D_k$ is also nowhere dense closed set (Indeed, if F_1 and F_2 are nowhere dense. If there is an open set $O \subset F_1 \bigcup F_2$, then $\emptyset \neq O \bigcap F_2^c = O \bigcap F_2^c \bigcap F_1 \subset F_1$, which is a contradiction.). Then $I_{n+1} \setminus \bigcup_{k=1}^{n} C_k \bigcup D_k$ is a nonempty open set which is a union of open intervals and thus contains a generalized cantor set C_{n+1} of positive measure. Then, in the same manner, we can also choose $D_{n+1} \subset I_{n+1} \setminus (\bigcup_{k=1}^{n} (C_k \bigcup D_k) \bigcup C_{n+1})$, which is of positive measure.

Now, define $A = \bigcup_n C_n$. By the construction, A works.

(2) We also restrict to $[0,1]$. Then for any subinterval $I \subset [0,1]$, $m(A \bigcap I) \leq 0.5m(I)$ implies that $m(A^c \bigcap I) \geq 0.5m(I)$. Then apply problem 1 in Jan 2016. □

Problem 1.9. Prove or disprove that the unit ball of $L^7(0,1)$ is closed in $L^1(0,1)$

Proof. It is right. Indeed, $B_1 = \{f : \|f\|_7 \leq 1\}$. Now, let $f_n \to f$ in L^1, where $f_n \in B_1$. Then, $f_n \to f$ in measure and thus there is a subsequence $f_{n_k} \to f$ a.e.(m). Then $|f_{n_k}|^7 \to |f|^7$ a.e.(m). Then, by Fatou's lemma, $\int |f|^7 \leq \liminf_k \int |f_{n_k}|^7 = 1$. □

Problem 1.10. Let C denote the Banach space of all convergent sequences under the norm $\|\cdot\|_\infty$. Compute the extreme points of the unit ball B of C and determined that whether B is the closed convex hull of its extreme points.

Proof. Fix a $a \in B$, if there is a m such that $|a(m)| < 1$. then there is a number δ such that $|a(m) - \delta| \leq 1$ and $|a(m) + \delta| \leq 1$. Now define $b_1, b_2 \in B$ such that $b_1(n) = a(n)$ whenever $n \neq m$ and $b_1(m) = a(m) + \delta$. In the same manner, define $b_2(n) = a(n)$ whenever $n \neq m$ and $b_1(m) = a(m) - \delta$. Then $a = (b_1 + b_2)/2$. Thus a is not an extreme point.

If for all n, $|a_n| = 1$. Let $a = (b_1 + b_2)/2$. Since $|b_i(n)| \leq 1$ for all n, $b_1(n) = b_2(n) = a(n)$, which implies that a is an extreme point. Thus $\mathrm{Ext}(B) = \{a : \forall n,\ |a(n)| = 1 \text{ and } \exists N, \forall n > N\ a(n) \equiv 1 \text{ or } a(n) \equiv -1.\}$

(Not sure)I think $\overline{\mathrm{conv}}(\mathrm{Ext}(B)) \neq B$. Let $e_0 = (1, 1, 1, 1, ...)$, $e_n = (0, ..0, 1, 0, ...)$ for $n \geq 1$. It can be seen that $\pm e_n \in \mathrm{conv}(\mathrm{Ext}(B))$ for all $n \geq 0$. Consider $\{e_n : n \geq 0\}$ form a basis of C. Then $\overline{\mathrm{conv}}(\mathrm{Ext}(B)) = B$ iff $\sum_{n=0}^{k} \alpha_n e_n \in \overline{\mathrm{conv}}(\mathrm{Ext}(B))$

for $1 \geq |\alpha_n| \to 0$. This may induce a contradiction of the convexity. Like consider $(1/n)$.

□

Problem 1.11. Show that every continuous convex function f defined on the closed unit ball of a reflexive Banach space X can achieve the minimum

Proof. At first, Since X is reflexive, then X is isomorphic to X^{**}. By Alaoglu's theorem, The unit ball of X^{**} is w^*-compact, which implies that the unit ball of X, B is weak compact. Then, we know that any function is lower semi-continuous convex iff it is weak lower-semi continuous convex. (This is a classical result in convex analysis. The epigraph is used in its proof or Mazur's lemma?) Thus f is weak-lower-semi-continuous. Thus it can achieve the minimum since B is weak compact. □

2. August 2016

Problem 2.1. Let $\mathcal{A}$ be the set of all real valued functions on $[0,1]$ for which $f(0) = 0$ and $|f(t) - f(s)| \leq (t-s)^2$ for $0 \leq s < t \leq 1$.
(1) Prove that $\mathcal{A}$ is a compact subset of $C[0,1]$.
(2) Prove that $\mathcal{A}$ is a compact subset of $L_1[0,1]$.

Proof. (1) At first, $\mathcal{A}$ is closed. Let $f_n \to f$ under $\|\cdot\|_\infty$. Then $|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)|$ implies that $\mathcal{A}$ is closed. It is also easy to check that $\mathcal{A}$ is equicontinuous and pointwise bounded. Then by Arzela-Ascoli theorem, $\mathcal{A}$ is compact in $C[0,1]$.

(2) Consider the identity map id from $C[0,1]$ to $L_1[0,1]$ is continuous by $\|f\|_1 = \int |f| \leq \|f\|_\infty$. Then $\mathcal{A}$ is compact in $L_1[0,1]$ since any continuous image of a compact set is also compact. □

Problem 2.2. (1) Let $f(x)$ be a real valued function on the real line that is differentiable almost everywhere. Prove that $f'(x)$ is a Lebesgue measurable function.
(2) If f is continuous real values function on the real line, then the set of points at which f is differentiable is measurable.

Proof. $f'(x) = \lim_{n\to\infty} n(f(x+1/n) - f(x))$ if the limit exists. Then Module a null set, f' is measurable. Thus, it is measurable.

For the second part, By the similar argument, We know that D^+f, D^-f, D_+f and D_-f are measurable. Then $\{x : f'(x) \text{ exsits}\} = \{x : D^+f(x) = D^-f(x) = D_+f(x) = D_-f(x)\}$ is measurable. □

Problem 2.3. (a) Let f be a real valued function on the unit interval $[0,1]$. Prove that the set of points at which f is discontinuous is a countable union of closed subsets.

(b) Prove that there is no real valued function on $[0,1]$ that is continuous at all rational points but discontinuous at all irrational points.

Proof. Define $\mathrm{osc}_f(x) = \inf\{\sup_{z,y\in U} |f(z) - f(y)| : U \text{ is a nbhd of } x\}$. It is not hard to see $\mathrm{osc}_f(x)$ is a continuous function and $\{x : f \text{ is continuous at } x\} = \{x : \mathrm{osc}_f(x) = 0\} = \bigcap_{n=1}^{\infty}\{x : \mathrm{osc}_f(x) < 1/n\}$, which is a G_δ set. It implies that $\{x : f \text{ is discontinuous at } x\}$ is a F_σ, i.e. a countable union of closed subsets.

For the second part, suppose there is a one. Then $\{x : f \text{ is continuous at } x\} = \mathbb{Q}$ is a dense G_δ set, say co-meager, which means that the set Ir of all irrationals is meager. Since $\mathbb{Q}$ is also countable, thus meager, $[0,1] = \mathbb{Q}\bigsqcup Ir$ is also a meager set. A contradiction to the Baire category theorem. □

Problem 2.4. Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and (f_n) be a sequence of measurable functions on X that converges pointwise to zero. Prove that (f_n) converges in measure to zero. Show that the converse is false for $[0,1]$ with Lebesgue measure.

Proof. Fix $\epsilon > 0$, $\delta > 0$ and define $A_n = \{x : |f_n(x)| \geq \epsilon\}$. For the δ, By Egoroff's theorem, there is a measurable set E with $\mu(E) < \delta$ and $f_n \to 0$ on E^c uniformly, say, there is a N, whenever $n > N$, $x \in E^c$, $|f_n(x)| < \epsilon$. It implies that $A_n \subset E$ and thus $\mu(A_n) < \delta$. It shows that $\mu(A_n) \to 0$, which means $f_n \to 0$ in measure.

For the second part. A counterexample is $f_n = \chi_{[j/2^k,(j+1)/2^k]}$ where $n = 2^k + j$ with $0 \leq j < 2^k$ and $k \in \mathbb{N}$. □

Problem 2.5. If f is Lebesgue integrable on the real line, prove that $\lim_{h\to 0}\int_{\mathbb{R}} |f(x+h) - f(x)|dx = 0$.

Proof. It suffices to show $\lim_{n\to\infty}\int_{\mathbb{R}} |f(x+h_n) - f(x)|dx = 0$ for every $(h_n) \to 0$. Suppose at first that f is continuous, then $g_n = |f(x+h_n) - f(x)| \to 0$ and $g_n \leq |f(x+h_n)| + |f(x)| \in L_1(\mathbb{R})$. Then DCT implies that $\lim_{n\to\infty}\int_{\mathbb{R}} |f(x+h_n) - f(x)|dx = 0$. Then in general, since $C_c(\mathbb{R})$ is dense in $L_1(\mathbb{R})$, there is a $p \in C_c(\mathbb{R})$ such that $\int_{\mathbb{R}} |f - p| < \epsilon/3$. Then by the inequality $|f(x+h_n) - f(x)| \leq |f(x+h_n) - p(x+h_n)| + |p(x+h_n) - p(x)| + |f(x) - p(x)|$, we can see that for the ϵ above, there is a N such that whenever $n > N$, $\int_{\mathbb{R}} |f(x+h_n) - f(x)|dx < \epsilon$. Then we are done. □

Problem 2.6. Prove or disprove that there is a sequence (P_n) of polynomials such that $(P_n(t))$ converges to one for every $t \in [0,1]$ but $\int_0^1 P_n(t)dt$ converge to two as $n \to \infty$

Proof. It is not hard to see there is a sequence of continuous functions (f_n) satisfying the statement. Like f_n is defined to be $\equiv 1$ on $[0, 1-1/n]$ and f_n be the piecewise linear function on $[1-1/n, 1]$ which are two line pass points (1-1/n,1), (1-1/2n,2n+1) and $(1,1)$. It satisfies that $f_n(t) \to 1$ for all $t \in [0,1]$ but $\int_0^1 f_n(t)dt \equiv 2$. Then for each n, applying Stone-Weirstrass theorem, we can find a polynomial P_n with $\|f_n - P_n\|_\infty < 2^{-n}$. It is not hard to check (P_n) works. □

Problem 2.7. Let (f_n) be a uniformly bounded sequence of continuous functions on $[0,1]$ that converges pointwise to zero. Prove that 0 is in the norm closure in $C[0,1]$ of the convex hull of (f_n).

Proof. By Geometrical version of Hahn-Banach theorem, $\overline{\text{conv}\{(f_n)\}}^w = \overline{\text{conv}\{(f_n)\}}^n$. Then, it suffice to verify that $0 \in \overline{\text{conv}\{(f_n)\}}^w$. Indeed, by Reisz's representation theorem, $C[0,1]^* = M[0,1]$. Then for all $\mu \in M[0,1]$, $|\int f_n d\mu| \le \int |f_n| d|\mu| \to 0$ by DCT since (f_n) are uniform bounded and converges pointwise to 0. Then $(f_n) \to 0$ in the weak topology and we are done. □

Problem 2.8. Assume that X is a reflexive Banach space and ϕ is a continuous linear functional on X. Prove that there is a norm one vector x such that $\phi(x) = \|\phi\|$. Give an counterexample in the case $X = l_1$.

Proof. By problem 11 in Jan 2017, we see the Ball B of X is weak compact. It is also easy to verify that ϕ is weak continuous since it is norm continuous and so is $|\phi|$. Then $|\phi|$ achieve the max value on B, say there is an element x such that $|\phi(x)| = \max_{y \in B} |\phi(y)| \ge \sup\{|\phi(y)| : \|y\| \le 1\} = \|\phi\|$. Thus $|\phi(x)| = \|\phi\|$ and $\|x\| = 1$ holds necessarily by $|\phi(x)| \le \|\phi\|\|x\|$. Then we can choose a number $e^{i\theta}$ such that $\phi(e^{i\theta}x) = |\phi(x)| = \|\phi\|$ and $\|e^{i\theta}x\| = 1$.

A counterexample: $l_1^* = l_\infty$. Then let $f = (1-1/n)_n \in l_\infty$. Then for every $x = (\alpha_n) \in l_1$ of norm 1, $|f(x)| = |\sum_n (1-1/n)\alpha_n| \le \sum_n (1-1/n)|\alpha_n| < \sum_n |\alpha_n| = 1 = \|f\|$. □

Problem 2.9. Suppose that X is a non separable Banach space. Prove that there is an uncountable subset A of the unit ball of X such that for all $x \neq y \in A$, $\|x - y\| > 0.9$.

Proof. Well, if you are familiar with set theory, you can define A by transfinite recursion by applying Reisz's lemma, i.e. Problem 5 in Jan 2016.

Indeed, we choose any norm one element x_0 to start. Assume we have defined $\{x_0, ..., x_\alpha\}$, where α is a countable ordinal i.e. $\alpha < \omega_1$ such that $x_\beta \neq x_\gamma$ implies that $\|x_\beta - y_\gamma\| > 0.9$. To define $x_{\alpha+1}$, Let $Y = \overline{\mathbb{Q} - \text{span}}\{x_\beta : 0 \leq \beta \leq \alpha\}$ is a proper Banach subspace of X since X is non separable. Then, by Problem 5(a) in Jan 2016, we can choose a $x_{\alpha+1}$ of norm 1 such that $\|x_{\alpha+1} - x_\beta\| > 0.9$ for all $\beta \leq \alpha$.

Now, If we have defined $\{x_0, ..., x_\beta : \beta < \alpha\}$ and $\alpha < \omega_1$ is a limit ordinal. To define x_α, in the same manner, firstly define $Y = \overline{\mathbb{Q} - \text{span}}\{x_\beta : 0 \leq \beta < \alpha\}$, which is a proper Banach subspace of X and we can pick up x_α of norm 1 such that $\|x_\alpha - x_\beta\| > 0.9$ for all $\beta < \alpha$.

Now let $A = \{x_\alpha : \alpha < \omega_1\}$ works. □

Problem 2.10. If A is a Borel subset of the line. Then $E = \{(x, y) : x - y \in A\}$ is a Borel subset of the plane. If $m(A) = 0$, then $m \times m(E) = 0$.

Proof. $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, y) = x - y$ is continuous. Thus, $E = f^{-1}(A)$ is Borel. $E^y = \{x \in \mathbb{R} : (x, y) \in E\} = y + A$ which is a null set since $m(y + A) = m(A) = 0$. Thus $m \times m(E) = \int m(E^y)dm(y) = 0$. □

3. January 2016

Problem 3.1. Let E be a measurable subset of $[0, 1]$. Suppose there exists $\alpha \in (0, 1)$ such that $m(E \bigcap J) \geq \alpha \cdot m(J)$ for all subintervals J of $[0, 1]$. Prove that $m(E) = 1$.

Proof. For any open subset U of $[0, 1]$, $U = \sqcup_{i=1}^{\infty} J_i$ for countable many open intervals J_i. It implies that $m(E \bigcap U) = \sum m(J_i \bigcap E) \geq \alpha \cdot \sum m(J_i) = m(U)$. So if $m(E) < 1$, say, $m(E^c) = a > 0$. For all $\epsilon > 0$, Let U be open such that $m(U - E^c) = m(U \cap E) < \epsilon$. But $\epsilon > m(U \cap E) \geq \alpha m(U) \geq \alpha a$. A contradiction. □

Problem 3.2. let $f, g \in L^1([0, 1])$. Suppose $\int_0^1 x^n f(x)dx = \int_0^1 x^n g(x)dx$ for all integers $n \geq 0$. Prove that $f = g$ a.e.

Proof. Let $h = f - g$. By the assumption, $\int_0^1 p(x)h(x)dx = 0$ for all polynomial $p(x)$ on $[0, 1]$. Then, by Stone-Weirstrass theorem, for all continuous function u on $[0, 1]$, $\int_0^1 uh = 0$. Now, suppose there is a measurable set E which is not a null set, such that $h \neq 0$ on E. Without loss of generality, we may assume $h > 0$ on E by replacing E with $E^+ = \{x \in E \colon h > 0\}$ or replacing h with $-h$ and E with $E^- = \{x \in E \colon h < 0\}$. We may also assume h is bounded on E, say, $h < m$ for some $m \in \mathbb{N}$. Indeed, since $h \in L_1$, $E_\infty = \{x \in E \colon h = \infty\}$ is null. Then consider $E = \bigcup_{m=1}^{\infty}\{x \in E\colon h < m\} \bigcup E_\infty$. There is a m such that E_m is not null and replace E by this E_m. It also implies that $\int_E h > 0$. Then, we know E can be approximated by a finite union of open intervals, say, for every $\epsilon > 0$, there is a $A = \bigsqcup_{i=1}^{n} I_i$ such that $\mu(E\Delta A) < \epsilon$. Thus, we have $\int_A h > 0$ since $|\int_E h - \int_A h| \leq \int_{E\Delta A} h \leq m\mu(E\Delta A) < m\epsilon$. Then, fix a continuous function u such that $u = 1$ on A. It implies that $\int_A uh > \int_A h > 0$. A contradiction. □

Problem 3.3. Let f, g

Problem 3.4. Let $\{g_n\}$ be a sequence of measurable functions on $[0, 1]$ such that (1) $|g_n(x)| \leq C$ for a.e.$x \in [0, 1]$ and (2) $\lim_{n\to\infty} \int_0^a g_n = 0$ for every $a \in [0, 1]$. Prove that for each $f \in L^1[0, 1]$, we have $\lim_{n\to\infty} \int_0^1 fg_n = 0$

Proof. By some standard approximation argument, we can see $S = \text{span}\{\chi_{[0,a]} \colon 0 \leq a \leq 1\}$ is dense in $C_c([0, 1])$ with respect to L_1-norm. Furthermore, since $C_c([0, 1])$ is also dense in $L_1[0, 1]$ with respect to L_1-norm, S is also dense in $L_1[0, 1]$ with respect to L_1-norm. Then, for every $f \in L_1[0, 1]$, there is a sequence $h_m = \sum_{i=1}^{K_m} k_i^{(m)} \chi_{[0,a_i]} \to f$. Then, by (2), for every m, $\lim_{n\to\infty} \int_0^1 h_m g_n = 0$. Then, for every ϵ, choose a m such that $\|h_m - f\|_1 < \epsilon$. Then for such m, there is a N such that whenever $n > N$, $|\int_0^1 h_m g_n| < \epsilon$. It implies that $|\int_0^1 fg_n| \leq |\int_0^1 (f - h_m)g_n| + |\int_0^1 h_m g_n| \leq C\|h_m - f\|_1 + \epsilon \leq (C + 1)\epsilon$. Then we are done. □

Problem 3.5. (a) Let X be a normed vector space and Y be a closed liner subspace of X. Assume Y is a proper subspace, that is, $Y \neq X$. Show that, for arbitrary $\epsilon \in (0, 1)$, there is an element $x \in X$ such that $\|x\| = 1$ and $\inf_{y\in Y} \|x - y\| > 1 - \epsilon$.

(b) Use part (a) to prove that, if X is an infinite dimensional normed vector space, then the unit ball of X is not compact.

Proof. For all ϵ, and a $x \notin Y$, $\inf_{y\in Y} \|x-y\| = d > 0$. Now, choose a $\delta > 0$ such that $\frac{d}{d+\delta} > 1-\epsilon$. For this δ, choose $y_0 \in Y$ such that $\|x-y_0\| < d+\delta$. Define $u = \frac{x-y_0}{\|x-y_0\|}$. Then $\|u\| = 1$ and $\|u + Y\| = \inf_{y\in Y} \|\frac{x-y_0}{\|x-y_0\|} - y\| = \frac{\|x+Y\|}{\|x-y_0\|} > \frac{d}{d+\delta} > 1 - \epsilon$.

If X is infinite dimensional, we can choose a sequence $\{x_n\}$ by induction in the unit ball. We begin with any element x_1 in the unit ball. Then If $\{x_1, x_2, ..., x_{n-1}\}$ has been defined, then by (a), there is an element x_n of norm 1 such that $\|x_n+Y\| > \frac{1}{2}$ where $Y = \text{span}\{x_1, ..., x_{n-1}\}$. Then $\{x_n\}$ witnesses that the unit ball is not compact since $\|x_n - x_m\| > \frac{1}{2}$ for all n, m. □

Problem 3.6. Let $\{f_k\}$ be a sequence of increasing functions on $[0, 1]$. Suppose $\sum_{k=1}^{\infty} f_k(x)$ converges for all $x \in [0, 1]$. Denote the limit function by f. Prove that $f'(x) = \sum_{k=1}^{\infty} f_k'(x)$ a.e. $x \in [0, 1]$.

Proof. $n(f(x+1/n)-f(x)) = n\sum_k (f_k(x+1/n)-f_k(x)) \geq n\sum_k \int_n^{x+1/n} f_k' = n\int_x^{x+1/n}\sum_k f_k'$ since each f_k' is positive. Then, we know that $\lim_{n\to\infty} n\int_x^{x+1/n}\sum_k f_k' = \sum_k f_k'(x)$, which implies that $f'(x) \geq \sum_k f_k'(x)$. In the converse, fix $x\in[0,1]$. Since f_k and f are increasing, then the points that f_k and f are not continuous are countable. Now, choose $h_n \downarrow 0$ and define $A = \{x, x+h_n : n\in\mathbb{N}\}$ on which f_k and f are continuous. A is closed and thus compact. Let $g_m(x) = \sum_k^m f_k(x)$. W.L.O.G, we may assume $f_k \geq 0$ by replacing f_k with $f_k - f_k(0)$. Then $g_m > 0$ and $g_m \uparrow f$ on A. Define $\sigma(h_n) = \frac{f(x+h_n)-f(x)}{h_n}$ and $\sigma_m(h_n) = \frac{g_m(x+h_n)-g_m(x)}{h_n}$, which are defined on $A - x = \{0, h_n : n\in N\}$, which is also compact. It can be verified that σ and all σ_m are continuous on $A-x$ and $\sigma_m \uparrow \sigma$ on $A-x$ and thus uniformly by Dini's theorem. It implies that for every ϵ, there is a M such that whenever $m > M$ and all $n\in\mathbb{N}$, $\sigma(h_n) < \sigma_m(h_n) + \epsilon$. Take a limit with respect to n, we have $f'(x) \leq g_m'(x) + \epsilon = \sum_k^m f_k'(x) + \epsilon \leq \sum_k^\infty f_k'(x) + \epsilon$. Thus, $f'(x) \leq \sum_k f_k'(x)$.

The following argument may be helpful to simplify the proof above but it lacks some uniform bound of $v(k,x)$ with respect to k now in order to apply DCT. $f_k'(x)$ is a good candidate but not good enough. See below.

Fix $x\in[0,1]$. Define $u(k,x) = f_k(x)$ and let δ be the counting measure on $\mathbb{N}$. Then $f(x) = \sum_k f_k(x) = \int_{\mathbb{N}} u(k,x)d\delta(k)$. Let $h_n \downarrow 0$. Also define $v_n(k,x) = \frac{u(k,x+h_n)-u(k,x)}{h_n}$. By definition, $v_n(k,x) \to f_k'(x)$. Furthermore, $\frac{f(x+h_n)-f(x)}{h_n} = \int_{\mathbb{N}} \frac{u(k,x+h_n)-u(k,x)}{h_n} d\delta(k) = \int_{\mathbb{N}} v_n(k,x)d\delta(k)$. It implies that $f'(x) = \lim_n \int_{\mathbb{N}} v_n(k,x)d\delta(k)$. If we can apply DCT, then we are done. □

Problem 3.7. Suppose $f, g\colon [0,1]\to\mathbb{R}$ are both continuous and of bounded variation. Show that the set $\{(f(t),g(t))\in\mathbb{R}^2 : t\in[0,1]\}$ cannot cover the entire unit square $[0,1]\times[0,1]$.

Proof. Define $r(t) = (f(t),g(t))$. Then since on $\mathbb{R}^2$, l_1^2 norm is equivalent to l_2^2 norm, r is a $\mathbb{R}^2$-valued function of BV, say whenever $0 = x_0 < x_1 < ... < x_n = b$, $\sum_{i=1}^n \|r(x_i) - r(x_{i-1})\|_2 < \infty$. Suppose $[0,1]\times[0,1]$ can be covered. Divide $[0,1]\times[0,1]$ into n^2-1 small squares, with center z_j, in which the length of each edge is $1/n$. Then, we can choose t_j such that $r(t_j) = z_j$ and reorder t_j in increasing order i.e. $s_1 < s_2 < ... < s_{n^2}$. Then, $\sum_{j=1}^{n^2-1} \|r(s_j) - r(s_{j+1})\|_2 \geq \sum_{j=1}^{n^2-1} 1/n = (n^2-1)/n = n - 1/n \to \infty$. A contradiction. □

Problem 3.8. (a) Suppose f is a measurable function on $[0,1]$, then $\|f\|_{L^\infty} = \lim_p \|f\|_{L^p}$.

(b) If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.

Proof. (a) By Hölder, assuming $1 \leq p < q < \infty$, we have $\|f\|_p^p = \int |f|^p \cdot 1 \leq \||f|^p\|_{\frac{q}{p}} \cdot \|1\|_{\frac{q}{q-p}} = \|f\|_q^p \cdot \mu([0,1])^{\frac{q-p}{q}}$, which implies that $\|f\|_p \leq \|f\|_q$. If $q=\infty$,

$\|f\|_p^p = \int |f|^p \leq \|f\|_\infty \cdot \mu([0,1])$. Thus, we see $\|f\|_p$ is increasing and bounded by $\|f\|_\infty$. for all ϵ, let $E = \{x\colon |f(x)| > \|f\|_\infty - \epsilon\}$. Then, $\|f\|_p^p \geq \int_E |f|^p \geq (\|f\|_\infty - \epsilon)^p \mu(E)$. Then, $\|f\|_p \geq \mu(E)^{\frac{1}{p}}(\|f\|_\infty - \epsilon)$. Now, let $p \to \infty$, we have $\lim \|f\|_p \geq \|f\|_\infty - \epsilon$, which implies that $\lim_p \|f\|_p = \|f\|_\infty$

(b) For the $\liminf_n \int f_n$, we can choose a sequence $\int f_{n_k}$ such that $\lim_k \int f_{n_k} = \liminf_n \int f_n$. Since $f_n \to f$ in measure, $f_{n_k} \to f$ in measure. Then there is a subsequence $f_{n_{k_m}}$ converging to f a.e. This implies that $\int f = \int \lim_m f_{n_{k_m}} \leq \lim_m \int f_{n_{k_m}} = \liminf_n \int f_n$ by Fatou's lemma. □

Problem 3.9. Suppose $\{f_n\}$ is a sequence of functions in $L^2[0,1]$ such that $\|f_n\|_2 \leq 1$. If $f_n \to f$ in measure, then

(a) $f \in L^2[0,1]$;

(b) $f_n \to f$ weakly in L^2;

(c) $f_n \to f$ w.r.t norm in L^p for $1 \leq p < 2$

Proof. (a) Since $f_n \to f$ in measure, there is a subsequence f_{n_k} converging to f a.e. Then $\int |f|^2 \leq \liminf \int |f_{n_k}|^2 \leq 1$.

(b) $f_n \to f$ in measure. Then, for all $h \in L^2[0,1]$, $f_n h \to f_n h$ in measure, thus cauchy in measure. Let $A_{m,n} = \{x\colon |f_n(x)h(x) - f_m(x)h(x)| \geq \epsilon\}$. Then, $\int_0^1 |f_n h - f_m h| = \int_{A_{m,n}} |f_n h - f_m h| + \int_{[0,1]\setminus A_{m,n}} |f_n h - f_m h| \leq \int_{A_{m,n}} |f_n h| + \int_{A_{m,n}} |f_m h| + \epsilon\mu([0,1]\setminus A_{m,n})$. Then, for all ϵ, there is a δ such that whenever $\mu(A) < \delta$, $\int_A |f_n h| \leq (\int_A |f_n|^2 \cdot \int_A |h|^2)^{\frac{1}{2}} \leq (\int_A |h|^2)^{\frac{1}{2}}$. Now, choose N big enough, such that $m, n > N$ implies that $\mu(A_{m,n}) < \delta$. Thus, $m, n > N$ also implies that $\int_0^1 |f_n h - f_m h| \leq 3\epsilon$. Thus, $f_n h$ is cauchy in $L^1[0,1]$, and thus converges to some g. Meanwhile, $f_n h \to g$ in measure, which implies that $g = fh$ and thus $|\int f_n h - \int f_h| \leq \int |f_n h - fh| \to 0$. Thus, $f_n \to f$ weakly in L^2.

(c) Define $E_n = \{x\colon |f_n(x) - f(x)| \geq \epsilon\}$. By the problem 1.8(a), $\|f_n\|_p \leq \|f_n\|_2 \leq 1$ and $\|f\|_p \leq \|f\|_2 \leq \infty$. Then, $\int_{E_n} |f_n - f|^p + \int_{E_n^c} |f_n - f|^p \leq 2^p(\int_{E_n} |f_n|^p + \int_{E_n} |f|^p) + \epsilon\mu(E_n^c)$. It remains to show that $\|f_n\|^p$ are uniformly integrable(see the Hint in the original problem). Indeed, by Hölder, $\int \|f_n\|^p \cdot \chi_A \leq \||f_n|^p\|_{\frac{2}{p}} \cdot \|\chi_A\|_{\frac{2}{2-p}} = \|f_n\|_2^p \cdot \mu(A)^{\frac{2}{2-p}} \leq \mu(A)^{\frac{2}{2-p}}$, which shows that $\||f_n|^p\|$ are uniformly integrable. □

Problem 3.10. Suppose E is a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = 99/100$. Show that there is a $x \in [0,1]$ such that for all $r \in (0,1)$, $m(E \bigcap (x - r, x + r)) \geq r/4$.

Proof. For any subset $A \subset [0,1]$, the Hardy-Littlewood Maximal function of χ_A is $M\chi_A(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_A = \sup_r \frac{1}{2r} m(A \bigcap (x - r, x + r))$. Now, suppose that the conclusion is not right, for every $x \in [0,1]$, there is a r_x such that $m(E \bigcap (x -$

$r_x, x + r_x)) \leq r_x/4$ i.e. $\frac{1}{2r_x} m(E \bigcap (x - r_x, x + r_x)) \leq 1/2$, which implies that $\frac{1}{2r_x} m(E^c \bigcap (x - r_x, x + r_x)) \geq 1/2$. Let $f = \chi_{E^c}$. Then, for every $x \in [0, 1]$, $Mf(x) \geq 1/2$. However, $m\{x : Mf(x) \geq 1/2\} \leq 6 \int \chi_{E^c} = 3/50$. A contradiction. □

4. August 2015

Problem 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. For each $t \in \mathbb{R}$, define $f_t(x) = f(t + x)$. Prove $f_t(x)$ is a Borel measurable function for each fixed t.

Proof. $f_t^{-1}((-\infty, a)) = \{x : x + t \in f^{-1}((-\infty, a))\} = f^{-1}((-\infty, a)) - t$ is Borel. □

Problem 4.2. justify the statement that $\int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dxdy = \int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dydx$.

Proof. To apply the Fubini's thm, it suffices to show $\int_0^1 \int_0^1 |\frac{(x-y)\sin(xy)}{x^2+y^2}| dxdy < \infty$. We integrate this on the quarter of a disk of radius $\sqrt{2}$ in the first quadrant, which contains $[0, 1] \times [0, 1]$. We see that $\int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} |\frac{r\cos(\theta) - r\sin(\theta)}{r^2}| r dr d\theta \leq 2 \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{2}} dr d\theta = \sqrt{2}\pi$. □

Problem 4.3. Assume that $\{f_n\}$ is a sequence in $C[0, 1]$. Show that:

(a) (f_n) converges weakly to 0 iff (f_n) is bounded in $C[0, 1]$ and $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in [0, 1]$.

(b) If (f_n) converges weakly in $C[0, 1]$, then it converges in norm in $L_p[0, 1]$ for all $1 \leq p < \infty$.

Proof. Consider $C[0, 1]^* = M[0, 1]$.

(a) If $f_n \to 0$ weakly, then, for all $\mu \in M[0, 1]$, $\int f_n d\mu \to 0$. In particular, $\mu = \delta_t$, $t \in [0, 1]$, implies that $f_n(t) \to 0$. If we view $f_n \in M[0, 1]^*$, then $f_n(\mu) = \mu(f_n) \to 0$. Then $\sup_n |f_n(\mu)| < \infty$, which $\sup_n \|f_n\| < \infty$ by Principle of uniform boundedness theorem. In the converse, by DCT, $|\int f_n d\mu| \leq \int |f_n| d|\mu| \to 0$.

(b) W.L.O.G, we may assume $f_n \to 0$ weakly, then by (a), $|f_n(t)|^p \to 0$ and $(|f_n|^p)$ is bounded. Then DCT implies that $f_n \to 0$ in L^p. □

Problem 4.4. Let A be a Lebesgue null set in $\mathbb{R}$. Prove that $B = \{e^x : x \in A\}$ is also a null set.

Proof. $f(x) = e^x$ is absolutely continuous on any interval $[a, b]$ since f is differentiable on $[a, b]$ and $|f(x) - f(y)| \leq M|x - y|$, where $M = e^b$ and $x, y \in [a, b]$. At first, assume that $A \subset [a.b]$. $A \subset O_n$, where O_n is a sequence of open sets such that $m(O_n) \to 0$. Let $O_n = \bigsqcup_{i=1}^{\infty} I_{i,n}$. Then $f(A) \subset f(O_n) = \bigsqcup_{i=1}^{\infty} f(I_{i,n})$, which implies that $m(f(A)) \leq \sum_{i=1}^{\infty} m(f(I_{i,n})) \leq M \sum_{i=1}^{\infty} m(I_{i,n}) = M \cdot m(O_n) \to 0$. Then $m(f(A)) = 0$. If A is not bounded. Define $A_n = A \bigcap [n, n + 1)$ for all $n \in \mathbb{Z}$. $A = \bigsqcup_n A_n$. By the argument above $m(f(A_n)) = 0$ and thus $m(f(A)) = \sum_n m(f(A_n)) = 0$. □

Problem 4.5. (b) Show that if f and g are absolutely continuous on $[a,b]$, then so does $f \cdot g$.

(c) Give an example to show that (b) is false if $[a,b]$ is replaced by $\mathbb{R}$.

Proof. (b) Since f, g are continuous on $[a,b]$, thus bounded by M and N, respectively. Then, for all ϵ there is a δ such that whenever $\sum_{i=1}^n |x_i - y_i| < \delta$, $\sum_{i=1}^n |f(x_i) - f(y_i)| < \epsilon/2N$ and $\sum_{i=1}^n |g(x_i) - g(y_i)| < \epsilon/2M$. Thus $\sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(y_i)| = \sum_{i=1}^n |f(x_i)g(x_i) - f(x_i)g(y_i) + f(x_i)g(y_i) - f(x_i)g(y_i)| \leq M \sum_{i=1}^n |g(x_i) - g(y_i)| + N \sum_{i=1}^n |f(x_i) - f(y_i)| = \epsilon$

(c) $f(x) = g(x) = x$ are absolutely continuous on $\mathbb{R}$. However, for all $\delta > 0$, choose disjoint intervals $I_i = (m_i, m_i + \delta_i)$ for $i = 1, 2, ..., n$ such that $\sum_i \delta_i = \delta$, $m_i < m_{i+1}$ and $m_1 \delta \geq 1/2$. then $\sum_{i=1}^n |f(m_i + \delta_i)g(m_i + \delta_i) - f(m_i)g(m_i)| = \sum_{i=1}^n |m_i^2 + 2m_i\delta_i + \delta_i^2 - m_i^2| \geq 2\sum_{i=1}^n m_i\delta_i \geq m_1\delta \geq 1/2$ □

Problem 4.6. Let X, Y be Banach spaces and $T\colon X \to Y$ be a one-to-one, bounded and linear operator for which the range $T(X)$ is closed in Y. Show that for each continuous linear functional ϕ on X there is a continuous linear functional ψ on Y, so that $\phi = \psi \circ T$.

Proof. By open mapping theorem, $\phi \circ T^{-1}$ is a well-defined linear bounded functional on $T(X)$. Then, by Hahn-Banach Thm, it can be extent to some ψ on Y, say, $y \in T(X)$ implies that $\phi \circ T^{-1}(y) = \psi(y)$. It implies that $\psi(T(x)) = \phi(x)$ for all $x \in X$. □

Problem 4.7. Derive Open Mapping Theorem from the Closed Graph Theorem.

Proof. Let $T\colon X \to Y$ surjective, continuous linear. Define the quotient map $T'\colon X/\ker(T) \to Y$ by $T'(x + \ker(T)) = T(x)$. We show that T' is an isomorphism. At first. $\|T'(x + \ker(T))\| = \|T(x+y)\| \leq \|T\|\|x+y\|$ for all $y \in \ker(T)$, which implies that $\|T'\| \leq \|T\|$. In the converse, consider T^{-1}. Let $T(x_n) \to T(x)$ and $x_n + \ker(T) \to y + \ker(T)$, say, $\|x_n - y + \ker(T)\| \to 0$. For all n, choose $z_n \in \ker(T)$ s.t. $\|x_n + z_n - y\| - 2^{-n} \leq \|x_n - y + \ker(T)\| \to 0$. Thus $T(x_n) = T(x_n + z_n) \to T(y)$. Then $T(x_n) \to T(x)$ implies that $T(x) = T(y)$ and thus $x + \ker(T) = y + \ker(T)$. By Closed Graph theorem, T' is an isomorphism, and $T = T'P$, where P is the projection from X to $X/\ker(T)$, which is an open map. Then, $T(O) = T'(P(O))$ is open for all open subset of X. □

Problem 4.8. Let Y be a closed subspace of a Banach space X, with norm $\|\cdot\|$. Let $|||\cdot|||$ be a norm on Y which is equivalent to $\|\cdot\|$, meaning that there is a $C \geq 1$ so that for all $y \in Y$:

$$\frac{1}{C}|||y||| \leq \|y\| \leq C|||y|||.$$

Let S be the set of all linear functionals $\psi\colon X \to \mathbb{R}$, so that
$|\psi(y)| \leq |||y||| $ for all $y \in Y$ and
$\|\psi(x)\| \leq \|x\|$ for all $x \in X$.
Prove the following statements:
(1) $||||a||||\colon = \sup_{\psi \in S} |\psi(x)|$ defines a norm on X.
(2) $||||y|||| = |||y|||$ for $y \in Y$.
(3) The norm $||||\cdot||||$ and $\|\cdot\|$ are equivalent on X.

Proof. (1) easy to verify.

(2) For $y \in Y$, by definition of $||||\cdot||||$, $||||y|||| \leq |||y|||$.
In the converse, choose a $\phi \in (Y, |||\cdot|||)^*$, s.t. $\phi(y) = |||y|||$ and norm of ϕ is 1. Thus for all $y \in Y$, $|\phi(y)| \leq |||y||| \leq C\|y\|$. Then, ϕ can be extent to whole X with the same norm, say for all $x \in X$, $|\phi(x)| \leq C\|x\|$. Then $\phi \in S$ and thus $||||y|||| \geq |||y|||$.

(3) By def of $||||\cdot||||$, $||||x|||| \leq C\|x\|$. In the converse, for all $x \in X$, by Hahn-Banach theorem, there is a ϕ s.t. $\phi(x) = \|x\|$, and $\|\phi\| = 1$. Define $\psi = \frac{1}{C}\phi$, which implies that $\psi(x) = \frac{1}{C}\|x\|$ and $\|\psi(z)\| \leq \frac{1}{C}\|z\| \leq C\|z\|$ for all $z \in X$ while $\|\psi(y)\| \leq \frac{1}{C}\|y\| \leq |||y|||$ for all $y \in Y$. Thus $\psi \in S$ and thus $||||x|||| \geq \frac{1}{C}\|x\|$. □

Problem 4.9. Let f be increasing on $[0,1]$ and let

$$g(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \qquad x \in (0,1).$$

Prove that if $A = \{x \in (0,1)\colon g(x) > 1\}$, then $f(1) - f(0) \geq m^*(A)$.

Proof. For all $x \in A$, for all $\delta > 0$, there is a h_δ s.t. $|h_\delta| < \delta$ and $\frac{f(x+h_\delta) - f(x-h_\delta)}{2h_\delta} > 1$. Now, we consider closed intervals I_x^δ centered at x of radius h_δ, then $\{I_x^\delta\colon x \in A, \delta > 0\}$ form a Vitali cover of A. By Vitali's Lemma. For $\epsilon > 0$, there is $\{I_{x_i}^{\delta_i}\}_{i=1}^n$ such that $\sum_{i=1}^n l(I_{x_i}^{\delta_i}) > m^*(A) - \epsilon$ and $I_{x_i}^{\delta_i} \bigcap I_{x_j}^{\delta_j} = \varnothing$. Then

$$f(1) - f(0) \geq \sum_{i=1}^n (f(x_i + h_{\delta_i}) - f(x_i - h_{\delta_i})) \geq \sum_{i=1}^n 2h_{\delta_i} = \sum_{i=1}^n l(I_{x_i}^{\delta_i}) > m^*(A) - \epsilon,$$

which implies that $f(1) - f(0) \geq m^*(A)$. □

Problem 4.10. Let A be a uniformly dense subspace of $C[0,1]$ and let $B = \{F(x)\colon F(x) = \int_0^x f(t)dt, 0 \leq x \leq 1, f \in A\}$. Prove B is uniformly dense in $C_0[0,1] = \{g \in C[0,1]\colon g(0) = 0\}$. And prove that the span of $\{\sin(nx)\colon n \in \mathbb{N}\}$ is dense in $C_0[0,1]$.

Proof. Let $B' = \{F(x)\colon F(x) = \int_0^x f(t)dt, 0 \leq x \leq 1, f \in C[0,1]\}$. Firstly, B is dense in B'. Indeed, Let $F \in B'$ and $G \in B$. $F(x) - G(x) = \int_0^x (f(t) - g(t))dt$ implies that $\|F - G\|_\infty \leq \int_0^1 |f(t) - g(t)|dt \leq \|f - g\|_\infty$. Then, since A is dense in $C[0,1]$, B is dense in B'.

B' is an algebra. Let $F, G \in B'$, say, $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$. Then, $F(x) \cdot G(x) = \int_0^x (F(t)g(t) + G(t)f(t))dt \in B'$. B' also separate points since $x = \int_0^x 1dt \in B'$. Then, by Stone-Weirstrass theorem, B' is dense in $C_0[0, 1]$ and thus so does B

$\sin(nx) = \int_0^x n\cos(nt)dt$. Then by the argument above, it suffices to show that $A = \text{span}\{n \cos(nt)\} = \text{span}\{\cos(nt)\}$ is dense in $C[0, 1]$. Indeed, $\cos(mt)\cos(nt) = \frac{1}{2}(\cos(|m - n|t) + \cos((m + n)t)) \in A$ and $1 = \cos(0 \cdot t) \in A$ separates points on $[0, 1]$. Then Stone-Weirstrass thm implies that A is dense in $C[0, 1]$. □

5. January 2015

Problem 5.1. Let $f \in L^1(\mathbb{R})$. If $\int_a^b f(x)dx = 0$ for all rational numbers $a < b$, prove that $f = 0$ a.e. in $\mathbb{R}$.

Proof. For all $c < d \in \mathbb{R}$, let $a_n \downarrow c$ and $b_n \uparrow d$, where $a_n, b_n \in \mathbb{Q}$. Define $f_n = f \cdot \chi_{[a_n,b_n]}$, then $f_n \to f \cdot \chi_{[c,d]}$. In addition, $|f_n| \leq |f| \in L^1$. Then, DCT implies that $\int_{[a_n,b_n]} f \to \int_{[c,d]} f = 0$. Then, if $\mu(\{x \colon f(x) \neq 0\}) > 0$, we may assume $\mu(\{x \colon f^+(x) \neq 0\}) > 0$ by $f = f^+$ or $f = -f^-$ whenever $f \neq 0$. Then, there is a $n \in \mathbb{N}$ such that $E_n = \{x \colon f^+(x) > 1/n\}$ is of positive measure $s > 0$. For $\epsilon = s/3n$, there is a δ_1 such that if $\mu(A) < \delta_1$, $|\int_A f d\mu| < \epsilon$. Let $\delta = \min\{s/3, \delta_1\}$, then there is an open set $A = \bigsqcup_1^m I_i$ such that $\mu(A \vartriangle E_n) < \delta$ and thus $\mu(A \bigcap E_n) \geq s - \delta$. Then, $\int_A f d\mu = \int_{A \bigcap E_n} f^+ d\mu + \int_{A \setminus E_n} f d\mu \geq \frac{1}{n}(s - \delta) - \epsilon \geq \frac{s}{3n} > 0$. A contradiction to $\int_A f = \sum_{i=1}^m \int_{I_i} f d\mu = 0$. □

Problem 5.2. Let $\{g_n\}$ and g be $L^1(\mathbb{R})$ and satisfy $\lim_{n\to\infty} \|g_n - g\|_1 = 0$. Prove that there is a subsequence of g_n that converges pointwise to g a.e.

Proof. Let $E_{\epsilon,n} = \{x \colon |g_n(x) - g(x)| > \epsilon\}$. Then $\epsilon\mu(E_{\epsilon,n}) \leq \int_{E_{\epsilon,n}} |g_n - g| \to 0$. Then for any $\epsilon > 0$, for all $\delta > 0$, there is a N such that whenever $n > N$, $\mu(E_{\epsilon,n}) < \delta$. Then, we choose $\{n_m\}$ by induction such that $\mu(F_m) < 2^{-m}$, where $F_m = \{x \colon |g_{n_m}(x) - g(x)| > 2^{-m}\}$. Now, define $D_k = \bigcup_{m=k}^\infty F_m$ and $D = \bigcap_{k=1}^\infty D_k$. Since $\mu(D_k) \leq \sum_{m=k}^\infty 2^{-m} \to 0$, $\mu(D) = 0$. Now, $x \in D^c$ iff there is a $k \in \mathbb{N}$, whenever $m \geq k$, $|g_{n_m}(x) - g(x)| < 2^{-m}$. Then we are done. □

Problem 5.3. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\mathcal{N} \subset \mathcal{M}$ be a σ-algebra. If $f \geq 0$ is $\mathcal{M}$-measurable and μ-integrable then prove that there exists an $\mathcal{N}$-measurable and μ-integrable function $g \geq 0$ so that

$$\int_E g d\mu = \int_E f d\mu, \qquad E \in \mathcal{N}.$$

Proof. Define $\nu(E) = \int_E f d\mu$ is a measure on $\mathcal{M}$. Then $\nu \ll \mu$. Then $\nu|_{\mathcal{N}} \ll \mu|_{\mathcal{N}}$. Then there is a g, which is $\mathcal{N}$-measurable, such that $\nu(E) = \int_E g d\nu$ by

Radon-Nikodym Theorem. In addition, $\nu(X) = \int_X f d\mu < \infty$ implies that g is μ-integrable. □

Problem 5.4. If H is a Hilbert space and $T \in L(H)$ satisfying that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in H$, then prove that T is bounded.

Proof. Let $x_n \to x$ and $Tx_n \to y$, we show that $Tx = y$. Indeed, for all $u \in H$. $|\langle x_n, Tu\rangle - \langle x, Tu\rangle| = |\langle x_n - x, Tu\rangle| \leq \|x_n - x\|\|Tu\| \to 0$ and $\langle Tx_n, u\rangle \to \langle y, u\rangle$ by the same argument. Thus $\langle Tx, u\rangle = \langle y, u\rangle$ for all $u \in H$. Then $\langle Tx-y, Tx-y\rangle = 0$, by replacing u by $Tx - y$, which implies that $Tx = y$. Then Closed Graph Thm implies that T is bounded. □

Problem 5.5. Let $f, g \in L^1(\mathbb{R})$. Prove that $h \in L^1(\mathbb{R})$, where $h(x) = \int_{\mathbb{R}} f(y)g(x-y)dy$ whenever this integral is finite.

Proof. $|h(x)| \leq \int_{\mathbb{R}} |f(y)||g(x-y)|dy$. Then, by Tonelli, $\int_{\mathbb{R}} |h(x)|dx \leq \int_{\mathbb{R}}(\int_{\mathbb{R}} |f(y)||g(x-y)|dy)dx = \int_{\mathbb{R}}(\int_{\mathbb{R}} |f(y)||g(x-y)|dx)dy = \int_{\mathbb{R}} |g| \cdot \int_{\mathbb{R}} |f| < \infty$ □

Problem 5.6. Let $f, g \in C[0, 1]$ with $f(x) < g(x)$ for all $x \in [0, 1]$.

(1) Prove there is a polynomial $p(x)$ so that

$$f(x) < p(x) < g(x), \qquad x \in [0, 1].$$

(2) Prove that there is an increasing sequence of polynomials $\{p_n(x)\}$ so that

$$f(x) < p_n(x) < g(x), \qquad x \in [0, 1]$$

and $p_n \to g$ uniformly on $[0, 1]$.

Proof. $g - f \in C[0, 1]$ and thus define $k = \min_{x\in[0,1]}(g(x) - f(x)) > 0$. Then, by Stone-Weirstrass Thm, there is a polynomial h such that $\|f - h\|_\infty < k/2$, which implies that for all $x \in [0, 1]$, $f(x) < h(x) + k/2$ and $g(x) - h(x) = g(x) - f(x) + f(x) - h(x) > k/2$, which implies that $f(x) < h(x) + k/2 < g(x)$ and define $p(x) = h(x) + k/2$.

For (2), define by induction, choose p_1 such that $f(x) < p_1(x) < g(x)$ and if $f(x) \leq g(x) - \frac{k}{2^n} \leq p_n(x) < g(x)$. Choose a p_{n+1} such that $g(x) > p_{n+1}(x) \geq \max\{g(x) - \frac{1}{2^{n+1}}, p_n(x)\}$ □

Problem 5.7. If $f \in L^2(\mathbb{R})$, $g \in L^3(\mathbb{R})$ and $h \in L^6(\mathbb{R})$, then prove that $fgh \in L^1(\mathbb{R})$

Proof. $|f(x)g(x)h(x)| \leq \frac{1}{2}|f(x)|^2 + \frac{1}{2}|g(x)h(x)|^2$, while $|g(x)h(x)|^2 \leq (\frac{2}{3}|g(x)|^3 + \frac{1}{3}|h(x)|^6)$. This implies that $|f(x)g(x)h(x)| \leq \frac{1}{2}|f(x)|^2 + \frac{1}{3}|g(x)|^3 + \frac{1}{6}|h(x)|^6$. Then, $\int |f(x)g(x)h(x)|dx \leq \frac{1}{2}\int |f(x)|^2dx + \frac{1}{3}\int |g(x)|^3dx + \frac{1}{6}\int |h(x)|^6dx < \infty$. □

Problem 5.8. (1) Y is metric space. Prove $y \in Y$ is isolated iff the complement $\{y\}^c$ is not dense in Y

(2) Let X be a countable nonempty complete metric space. Prove that the set of isolated points is dense in X.

Proof. If $\{y\}$ is open, then $\{y\} \bigcap \{y\}^c = \emptyset$, which implies that $\{y\}^c$ is not dense. In the converse, $\{y\}^c$ being not dense implies that there is an open set O such that $\{y\}^c \bigcap O = \emptyset$. Then $\{y\} = O$ and thus y is an isolated point.

For (2), If not, there is an open set O such that $O \bigcap \{y \in X : y\text{is isolated}\} = \emptyset$. It is not hard to see since X is complete, O itself is a Baire space, say, given a sequence $O_n \subset O$, in which memebers O_n are open and dense in O, then $\bigcap_n O_n$ is also dense in O (consider $U_n = O_n \bigcup \overline{O}^c$). Then, for all $y \in O$, y is not isolated. It implies that $O \setminus \{y\}$ is dense in O. Then by Baire category theorem, $\emptyset = \bigcap_{y \in O}(O \setminus \{y\})$ is also dense in O. A contradiction. □

Problem 5.9. Suppose $f \in L^p(\mathbb{R})$ for all $p \in (1,2)$ and that $\sup_{p\in(1,2)} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that $\lim_{p\to 2^-} \|f\|_p = \|f\|_2$.

Proof. $\|f\|_p^p = \int_E |f|^p + \int_{E^c} |f|^p$, where $E^c = \{x : |f(x)| \leq 1\}$. Now, for all increasing sequence $\{p_n > 1\} \uparrow 2$. On E, $|f(x)|^{p_n} \downarrow |f(x)|^2$. In addition, on E^c, $|f(x)|^{p_n} \uparrow |f(x)|^2$. Then, MCT implies that $\int_E |f|^{p_n} + \int_{E^c} |f|^{p_n} \to \int_{\mathbb{R}} |f|^2$, which implies that $\|f\|_{p_n}^{\frac{p_n}{2}} \to \|f\|_2$. Now, since $M = \sup_{p\in(1,2)} \|f\|_p < \infty$, then $\|f\|_{p_n}^{\frac{p_n}{2}-1} \leq M^{\frac{p_n}{2}-1} \to 1$, which implies that $\|f\|_{p_n}^{\frac{p_n}{2}} - \|f\|_{p_n} \to 0$ and thus $\|f\|_{p_n} \to \|f\|_2$.

Since $\{p_n\}$ is arbitrary, $\lim_{p\to 2^-} \|f\|_p = \|f\|_2$ holds and $\|f\|_2 < \infty$ since $M = \sup_{p\in(1,2)} \|f\|_p < \infty$. □

Problem 5.10. This problem is same with Problem 8 in Aug 2015

6. August 2014

Problem 6.1. For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be continuous, and for every $x \in [0,1]$ the sequence $(f_n(x))$ is decreasing. Suppose that $f_n \to f$ pointwise. Show the convergence is uniform.

Proof. This is Dini's theorem. Given an $\epsilon > 0$.Consider open sets $U_m = \{x : |f_m(x) - f(x)| < \epsilon\} = \{x : \forall x \geq M, |f_m(x) - f(x)| < \epsilon\}$ since f_m is monotone. Since $f_m \to f$ pointwise, $[0,1] = \bigcup_{m=1}^{\infty} U_m$. Then by the compactness of $[0,1]$, $[0,1] = \bigcup_{n=1}^{N} U_{m_n}$. Now, let $M = \max\{m_1, ..., m_N\}$. Then whenever $n > M$, for all $x \in [0,1]$, $|f_m(x) - f(x)| < \epsilon$. □

Problem 6.2. Let $f \in L^1(0,\infty)$. For $x > 0$, define $g(x) = \int_0^{\infty} f(t)e^{-tx}dt$. Prove that $g(x)$ is differentiable for $x > 0$ with derivative $g'(x) = \int_0^{\infty} -tf(t)e^{-tx}dt$

Proof. Define $h(x) = \int_0^x \int_0^\infty -tf(t)e^{-ty}dtdy$. Then $\int_0^x \int_0^\infty |-tf(t)e^{-ty}|dtdy = \int_0^\infty (\int_0^x e^{-ty}dy) \cdot |-tf(t)|dt = \int_0^\infty |-tf(t)| \cdot (-\frac{1}{t}e^{-tx} + \frac{1}{t})dt = \int_0^\infty -|f(t)|e^{-tx}dt + \int_0^\infty |f(t)|dt \leq 2\|f\|_1$. Thus, by Fubini, $h(x) = \int_0^\infty (\int_0^x e^{-ty}dy) \cdot (-tf(t))dt = \int_0^\infty (-tf(t)) \cdot (-\frac{1}{t}e^{-tx} + \frac{1}{t})dt = \int_0^\infty f(t)e^{-tx}dt - \int_0^\infty f(t)dt$. Then, $g = h + \int_0^\infty f(t)dt$. We are done. □

Problem 6.3. This problem is same with Problem 1 in Jan 2015.

Problem 6.4. Let f be Lebesgue measurable on $[0,1]$ with $f > 0$ a.e. Suppose E_k is a sequence of measurable sets in $[0,1]$ with the property that $\int_{E_k} f(x)dx \to 0$ as $k \to \infty$. Prove that $\mu(E_k) \to 0$ as $k \to \infty$.

Proof. Define $F_n = \{x : f(x) > 1/n\}$. $\frac{1}{n}\mu(F_n \bigcap E_k) \leq \int_{F_n \bigcap E_k} f \leq \int_{E_k} f \to 0$ as $k \to \infty$. Since F_n are increasing to the whole space, $\mu(E_k) = \lim_{n\to\infty} \mu(E_k \bigcap F_n)$ uniformly. Indeed, given an ϵ, there is a N such that whenever $n \geq N$, for all k, $|\mu(E_k \bigcap F_n) - \mu(E_k)| \leq \mu(F_n^c) < \epsilon/2$. For such N, there is a K, whenever $k > K$ $\mu(F_N \bigcap E_k) \leq N \int_{E_k} f \leq \epsilon/2$, which implies $\mu(E_k) < \epsilon$. Thus, $\lim_{k\to\infty} \mu(E_k) = 0$. □

Problem 6.5. Let (f_n) be a sequence of continuous functions on $[0,1]$ such that for each $x \in [0,1]$ there is an N_x so that $f_n(x) \geq 0$ for all $n \geq N_x$. Show that there is an open nonempty set $U \subset [0,1]$ and an $N \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq N$ and all $x \in U$.

Proof. If not, for all open set U, integer N, there is a $n > N$ and a point $x \in U$, such that $f_n(x) < 0$. Consider open sets $E_n = \{x : \exists m \geq n,\ f(x) < 0\}$ are dense since $E_n \bigcap U \neq \emptyset$. However, $\bigcap_n E_n = \emptyset$. A contradiction to Baire category theorem. □

Problem 6.6. Let X be an infinite dimensional Banach space. What is the w^*-closure of $S_{X^*} = \{x^* : \|x^*\| = 1\}$. (The best exercise to use Hahn-Banach and Krein-Milman theorem I have ever seen)

Proof. We claim that $\overline{S_{X^*}}$ in the w^* topology is Ball_{X^*}. Indeed, at first, if $\|x^*\| > 1$, then there is a $x \in X$ such that $\|x\| = 1$ and $|x^*(x)| > 1$. Then, there is an ϵ such that the nbhd of x^*, $A = \{y^* : |x^*(x) - y^*(x)| < \epsilon\}$ does not intersect with Ball_{X^*}. To see this, for all $y^* \in A$, $|y^*(x)| > 1$ and thus $\|y^*\| > 1$.

Then, fix a x^* with $\|x^*\| \leq 1$. Consider a general nbhd of x^*, say $O = \bigcap_{i=1}^n \{y^* : |x^*(x_i) - y^*(x_i)| < \epsilon\}$. Let $M = \text{span}\{x_i : i = 1, ..., n\}$. To simplify the notation, we denote ϕ for x^*. Let $H_\phi = \{f \in X^* : f|_M = \phi\}$. We claim that H_ψ is weak*-closed, convex and nonempty set.

Indeed, any Hahn-Banach extension of ϕ, say, f with $\|f\| = \|\psi\| \leq 1$ implies that H_ϕ is nonempty. For all $f_1, f_2 \in H_\phi$, $0 < \lambda < 1$, $\lambda f_1 + (1-\lambda)f_2|_M = \phi$ and

$\|\lambda f_1 + (1-\lambda) f_2\| \leq 1$. Thus, H_ϕ is convex. Now, let $f_\mu(x) \to f(x)$ for all $x \in X$, where $f_\mu \in H_\phi$. Then since $f_\mu|_M = \phi$, $f|_M = \phi$. In addition, for all x is of norm 1, $|f_\mu(x)| \leq \|f_\mu\|\|x\| \leq 1$. This implies that $|f(x)| \leq 1$ and thus $\|f\| \leq 1$, which implies that H_ϕ is w^*-closed.

Then, by Krein-Milman theorem, $\mathrm{Ext}(H_\phi) \neq \emptyset$. Let $f \in \mathrm{Ext}(H_\phi)$, we claim $\|f\| = 1$. Indeed, suppose $\|f\| < 1$. Let g be a linear functional, such that $g|_M = 0$ but $\|g\| = 1$. Then define $f_1 = f + (1-\|f\|)g$ and $f_1 = f - (1-\|f\|)g$. Then, it can be verified that $\|f_1\| \leq 1$, $\|f_2\| \leq 1$ and $f_1|_M = f_2|_M = \phi$. However, $f = (f_1+f_2)/2$. This contradicts to $f \in \mathrm{Ext}(H_\phi)$. Thus, $f \in S_{X^*} \bigcap O$. We are done. □

Problem 6.7. Let μ be a finite measure on the measurable space (Ω, Σ). Prove the following part of the proof of the above Theorem: If $F \in L_p^*(\mu)$, then there exists an $h \in L_1(\mu)$ so that $F(\chi_A) = \int_A h d\mu$ for all $A \in \Sigma$.

Proof. $F(\chi_A)$ is a measure on (Ω, Σ) such that $F(\chi_A) \ll \mu$. Indeed, $F(\chi_\emptyset) = F(0) = 0$. Let $A = \bigsqcup_{n=1}^\infty A_n$. $\chi_A = \sum_{n=1}^\infty \chi_{A_n}$ implies that $\sum_{n=1}^m \chi_{A_n} \to \chi_A$ in $L^p(\mu)$. Then since $F \in L^p(\mu)^*$, $F(\sum_{n=1}^m \chi_{A_n}) = \sum_{n=1}^m F(\chi_{A_n}) \to F(\chi_A)$, which implies that $F(\chi_A) = \sum_{n=1}^\infty F(\chi_{A_n})$. Thus, $F(\chi_A)$ is a measure. If $\mu(A) = 0$, $|F(\chi_A)| \leq K\|\chi_A\|_p = K\mu(A)^p = 0$. Then, Radon-Nikodym theorem applies. □

Problem 6.8. Assume that (x_n) is a weakly converging sequence in a Hilbert space H. Show that there is a subsequence (y_n) of (x_n) so that $\frac{1}{n}\sum_{j=1}^n y_j$ converges in norm.

Proof. (x_n) is bounded. W.L.O.G, we may assume $(x_n) \to 0$ by subtract its' limit. It allows to choose y_j by induction such that $|\langle y_j, \sum_{k=1}^{j-1} y_k\rangle| < 2^{-j}$. Now, for $n > m$, $\|\frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\| = \langle \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j, \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\rangle =$
$\langle (\frac{1}{m} - \frac{1}{n})\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j, (\frac{1}{m} - \frac{1}{n})\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j\rangle \qquad (\star)$

Let $\epsilon > 0$, by the choice of y_j, there is a $m \in N$, whenever $n \geq m$, $|(\frac{1}{m} - \frac{1}{n})\sum_{j=1}^m y_j, \frac{1}{n}\sum_{j=m+1}^n y_j\rangle| < \epsilon^2$. Then $(\star) \leq (\frac{1}{m} - \frac{1}{n})^2\|\sum_{j=1}^m y_j\|^2 + 2\epsilon^2 + \frac{1}{n^2}\|\sum_{j=m+1}^n y_j\|^2 \leq \frac{1}{m^2}(\sum_{j=1}^n \|y_j\|^2 + 2) + \frac{1}{n^2}(\sum_{j=m+1}^n \|y_j\|^2 + 2) + 2\epsilon^2 \leq \frac{1}{m^2}(m \cdot \sup_{j\in\mathbb{N}} \|y_j\|^2 + 2) + \frac{1}{n^2}(n \cdot \sup_{j\in\mathbb{N}} \|y_j\|^2 + 2) + 2\epsilon^2$ □

Problem 6.9. Show that a linear functional ϕ on a Banach space X is continuous iff $\{x \colon \phi(2x) = 3\}$ is norm closed.

Proof. $\Longrightarrow$ is trivial. In the converse. $\{x \colon \phi(2x) = 3\} = 3/2 + \ker(\phi)$. Since the shift by $3/2$ is a homeomorphism, then $\{x \colon \phi(2x) = 3\}$ is closed iff $\ker(\phi)$ is closed. Define $\phi' : X/\ker(\phi) \to \mathbb{C}$ by $\phi'(x + \ker(\phi)) = \phi(x)$, which is an isomorphism. Indeed, fix a point $x_0 \in X$, s.t. $\phi(x_0) = r \neq 0$. Then for all $w \in \mathbb{C}$, $\phi'(\frac{w}{r}x_0 + \ker(\phi)) = w$. Injectivity follows from the definition of the quotient map ϕ'. Thus, ϕ' is an isomorphism since $\dim(\mathbb{C}) = 1$. Thus, $\phi = \phi' \circ P$ is continuous since projection P is also continuous. ☐

Problem 6.10. Define $T : C^1[0,1] \to C[0,1]$ by $Tf = f'$. Show that T has closed graph and that T is not bounded.

Proof. Let $f(x) = \int_0^x f'(t)dt + M$ and $f_n(x) = \int_0^x f_n'(t)dt + M_n$. Let $f_n \to f$ and $f_n' \to g$ under $\|\cdot\|_\infty$. Since $f_n(0) \to f(0)$, $M_n \to M$. Then, $f_n' \to g$ implies that $f_n(x) = \int_0^x f_n'(t)dt + M_n \to \int_0^x g(t)dt + M$ under $\|\cdot\|_\infty$. Thus, $g = f'$.

On the other hand, let $f_n(x) = x^n$. $\|f_n\|_\infty = 1$ while $\|f_n'\|_\infty = n$. Thus, T is unbounded and $C_1[0,1]$ is not a Banach space under the uniform norm. ☐

7. January 2014

Problem 7.1. Let $(X, \mathcal{M}, \mu)$ be a non atomic measure space with $\mu(X) > 0$. Show that there is a measurable $f : X \to [0, \infty)$, for which $\int_X f d\mu = \infty$.

Proof. μ is called atomic if there is a $A \in \mathcal{M}$ such that $\mu(A) > 0$ such that whenever $B \subset A$ with $\mu(B) < \mu(A)$, $\mu(B) = 0$. Then, if μ is not atomic, we can define a decreasing sequence $X = E_1 \supset E_2 \supset ....$ such that $\mu(E_1) > \mu(E_2) > \mu(E_3) > ... > 0$. Thus define $f(x) = \frac{1}{\mu(E_n \setminus E_{n+1})} > 0$ if $x \in E_n \setminus E_{n+1}$ and $f(x) = 0$ if $x \in \bigcap_n E_n$. Then $\int_X f d\mu = \infty$. ☐

Problem 7.2. Assume that μ is a finite measure on $\mathbb{R}^n$. Prove that there is a closed set $A \subset \mathbb{R}^n$ with the property that for each closed $B \subsetneq A$ it follows that $\mu(A \setminus B) \neq 0$.

Proof. μ is Radon since it is finite. Now, let $M = \{U : U \text{ is open and } \mu(U) = 0\}$ and $O = \bigcup\{U : U \in M\}$. For any compact set $K \subset O$, there is a subcover i.e. $K \subset \bigcup_{i=1}^n U_i$, where all $U_i \in M$. It implies that $\mu(K) = 0$ and thus $\mu(O) = 0$ by the regularity of a Radon measure. Define $A = \mathbb{R}^n \setminus O$. Then for all closed set $B \subsetneq A$, B^c is open and $\mu(B^c) > 0$ by definition of A. Then $\mu(A \setminus B) = \mu(B^c) > 0$ since $\mu(O \setminus B) = 0$. ☐

Problem 7.3. For a nonnegative function $f \in L_1[0,1]$, prove that $\lim_{n\to\infty} \int_0^1 \sqrt[n]{f(x)}dx = m\{x : f(x) > 0\}$.

Proof. WLOG, We may assume $f > 0$ everywhere. Let $F = \{x : f(x) \geq 1\}$. $\int_0^1 f(x)^{1/n}dx = \int_F f(x)^{1/n}dx + \int_{F^c} f(x)^{1/n}dx$. On F, $f(x)^{1/n} \downarrow 1$. Similarly, On F^c, $f(x)^{1/n} \uparrow 1$. Then $\int_F f(x)^{1/n}dx \to m(F)$ and $\int_{F^c} f(x)^{1/n}dx \to m(F^c)$ by MCT and thus $\int_0^1 f(x)^{1/n}dx \to m\{x : f(x) > 0\}$. ☐

Problem 7.4. Let f be Lebesgue integrable on $(0,1)$. For $0<x<1$ define $g(x)=\int_x^1 t^{-1}f(t)dt$. Prove that g is Lebesgue integrable on $(0,1)$ and that $\int_0^1 g(x)dx=\int_0^1 f(x)dx$.

Proof. $\int_0^1 |g(x)|dx \le \int_0^1\int_x^1 |t^{-1}f(t)|dtdx = \int_0^1(\int_0^t |t^{-1}f(t)|dx)dt = \int_0^1 |f(t)|dt < \infty$ by Tonelli. Then Fubini is applied here to see $\int_0^1 g(x)dx = \int_0^1 f(x)dx$ with the same calculation. Note that the integration area is the upper triangle of the unit square of x-t axis. i.e. the triangle constructed by lines $x=t$, $x=1$ and $t=0$. □

Problem 7.5. Assume that ν and μ are two finite measures on a measurable space $(X,\mathcal{M})$. Prove that $\nu \ll \mu$ iff $\lim_{n\to\infty}(\nu - n\mu)^+ = 0$.

Proof. $(\Leftarrow)$: If $\mu(E)=0$. Then $(\nu-n\mu)(E)=\nu(E)$ and then $\lim_{n\to\infty}(\nu-n\mu)^+(E)=\nu(E)=0$, which implies that $\nu\ll\mu$.

$(\Rightarrow)$: If $\nu\ll\mu$ holds, then $\nu(E)=\int_E f d\mu$ for some positive μ-integrable function f. Then $(\nu-n\mu)(E)=\int_E(f-n)d\mu$, which implies that $(\nu-n\mu)^+(E)=\int_E(f-n)^+d\mu$. Since $(f(x)-n)^+\downarrow 0$ for all $x\in X$. Then MCT implies that $\lim_{n\to\infty}(\nu-n\mu)^+=0$. □

Problem 7.6. Let (p_n) be a sequence of polynomials which converges uniformly on $[0,1]$ to some function f, and assume that f is not a polynomial. Prove that $\lim_{n\to\infty}\deg(p_n)=\infty$.

Proof. If not, say, suppose $\max\{\deg(p_n)\}=m$. $X=\{p: p \text{ is a polynomial and } \deg(p)\le m\}$ is a finite dimensional liner subspace of $C[0,1]$, thus closed, which implies that $f\in X$ is a polynomial. A contradiction. □

Problem 7.7. Let (f_n) be sequence of non zero bounded linear functionals on a Banach space X. Show that there is an $x\in X$ so that $f_n(x)\neq 0$ for all $n\in\mathbb{N}$.

Proof. For each n, $\ker(f_n)$ is nowhere dense. Indeed, suppose there is a $B(x,\epsilon)\subset\ker(f_n)$. Then the open Ball of radius ϵ, $\mathrm{Ball}(\epsilon)\subset\ker(f_n)$, which implies that $f_n\equiv 0$. Thus, if the statement does not hold, $X=\bigcup_n\ker(f_n)$, which is a contradiction to Baire category theorem. □

Problem 7.8. Assume that $T: l_1\to l_2$ is bounded, linear and one-to-one. Prove that $T(l_1)$ is not closed in l_2.

Proof. If $T(l_1)$ is closed, then it is a Hilbert space and thus l_1 is also a Hilbert space by open mapping theorem, say T is in fact an isomorphism. A contradiction. □

Problem 7.9. This is same to Problem 3 in Angust 2015.

Problem 7.10. Assume that f is a measurable and non negative function on $[0,1]^2$ and that $1 \leq r < p < \infty$. Show that $(\int_0^1 (\int_0^1 f^r(x,y)dy)^{p/r} dx)^{1/p} \leq (\int_0^1 (\int_0^1 f^p(x,y)dx)^{r/p} dy)^{1/r}$.

Proof. Define $F(x) = \int_0^1 f^r(x,y)dy$ is a non negative function, $s = p/r$ and s' be the conjugate of s. Then for $h \in L_{s'}[0,1]$ with $\|h\|_{s'} = 1$, $Fh \in L_1[0,1]$ by Hölder. By Tonelli's theorem, $\int_0^1 \int_0^1 |f^r(x,y)h(x)|dydx = \int_0^1 \int_0^1 f^r(x,y)|h(x)|dydx = \int_0^1 F(x)|h(x)|dx < \infty$. Then $|\int_0^1 F(x)h(x)dx| = |\int_0^1 (\int_0^1 f^r(x,y)dy)h(x)dx| = |\int_0^1 (\int_0^1 f^r(x,y)h(x)dx)dy| \leq \int_0^1 \|f^r(\cdot,y)\|_s \|h\|_{s'} dy = \int_0^1 (\int_0^1 f^p(x,y)dx)^{r/p} dy$ by Fubini and Hölder.

Then, $\|F\|_s = \sup\{|\int_0^1 F(x)h(x)dx| : \|h\|_{s'} = 1\} \leq \int_0^1 (\int_0^1 f^p(x,y)dx)^{r/p} dy$. Then we are done since $\|F\|_s = (\int_0^1 (\int_0^1 f^r(x,y)dy)^{p/r} dx)^{r/p}$. □

8. August 2013

Problem 8.1. Let $1 \leq p \leq \infty$ and $f \in L^p(\mathbb{R})$. For $t \in \mathbb{R}$, let $f_t(x) = f(x-t)$ and consider the mapping $G : \mathbb{R} \to L^p(\mathbb{R})$ given by $G(t) = f_t$. The space $L^p(\mathbb{R})$ is equipped with the usual norm topology. (a) Show that G is continuous if $1 \leq p < \infty$. (b) Find an f for which the mapping G is not continuous when $p = \infty$. (c) Let $1 \leq p, q \leq \infty$ be conjugate exponents(i.e $1/p + 1/q = 1$). Let $f \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$. Show that $h = f * g$ is continuous, where $h(t) = \int_{\mathbb{R}} f(t)g(t-x)dx$.

Proof. The continuity of G when $1 \leq p < \infty$ share a same proof of Problem 5 in August 2016. If $p = \infty$, Consider $f = \chi_{[0,1)}$. Then for any $t_n \to 0$, $\|\chi_{[0,1),t_n} - \chi_{[0,1)}\| \equiv 1$.

For the last statement, define $g_t(x) = f(t-x)$. $|h(t) - h(t_n)| \leq \int_{\mathbb{R}} |f(x)||g(t-x) - g(t_n - x)|dx \leq \|f\|_p \|g_{t_n} - g_t\|_q$ by Hölder. By the same argument of the first part, if $t_n \to t$, then $\|g_{t_n} - g_t\|_q \to 0$. □

Problem 8.2. (a) For $f \in C_{\mathbb{R}}[0,1]$, show that $f \geq 0$ iff $\|\lambda - f\|_\infty \leq \lambda$ for all $\lambda \geq \|f\|_\infty$

(b) Suppose $E \subset C_{\mathbb{R}}[0,1]$ is a closed subspace containing the constant function 1. For $\phi \in E^*$, show $\phi \geq 0$ iff $\|\phi\| = \phi(1)$.

(c) If $\phi \in E^*$ and $\phi \geq 0$, show that there is a bounded linear functional ψ on $C_{\mathbb{R}}[0,1]$ so that $\psi \geq 0$ and the restriction of ψ to E is ϕ.

Proof. (a) ($\Rightarrow$) suppose that $f \geq 0$ and $\lambda \geq \|f\|_\infty$, then $0 \leq \lambda - f \leq \lambda$, whence $\|\lambda - f\|_\infty \leq \lambda$. ($\Leftarrow$) If there is a $x \in [0,1]$ such that $f(x) < 0$, which entails that $\lambda - f(x) > \lambda$ and thus $\|\lambda - f\|_\infty > \lambda$.

(b)($\Rightarrow$) $|f| \leq 1$ implies that $1 \pm f \geq 0$ and thus $\phi(1 \pm f) \geq 0$ since $\phi \geq 0$. Thus $|\phi(f)| \leq \phi(1)$, whence $\phi(1) \geq \|\phi\|$ and thus $\phi(1) = \|\phi\|$. ($\Leftarrow$) fix a $f \geq 0$.

$\phi(\|f\|_\infty \cdot 1 - f) \le \phi(1)\|\|f\|_\infty - f\| \le \phi(1)\|f\|_\infty = \phi(\|f\|_\infty)$ by part (a). Thus $\phi(f) \ge 0$

(c)By Hahn-Banach thm, there is an extension ψ of ϕ with the same norm, which implies that $\|\psi\| = \|\phi\| = \phi(1) = \psi(1)$. Thus $\psi \ge 0$ by part (b). □

Problem 8.3. (a) Let μ and λ be mutually singular complex measures defined on the same measurable space $(X, \mathcal{M})$ and let $\nu = \mu + \lambda$. Show that $|\nu| = |\mu| + |\lambda|$.

(b) Construct a nonzero atomless Borel measure on $[0, 1]$ that this mutually singular with respect to Lebesgue measure m.

Proof. (a) Let F is null for λ while E is null for μ, with $E \bigsqcup F = X$. Let $P_2 \bigcup N_2 = F$ be a Hahn decomposition for λ while $P_3 \bigcup N_3 = E$ be a Hahn decomposition for μ. Then $P_1 = P_2 \bigcup P_3$, with $N_1 = N_2 \bigcup N_3$ is a Hahn decomposition for ν. Then $\nu^+(A) = \nu(A \bigcap P_2) + \nu(A \bigcap P_3) = \mu(A \bigcap P_2) + \lambda(A \bigcap P_3)$ and similarly, $\nu^-(A) = \nu(A \bigcap N_2) + \nu(A \bigcap N_3) = \mu(A \bigcap N_2) + \lambda(A \bigcap N_3)$, which implies that $|\nu| = |\mu| + |\lambda|$.

(b) Let f be the cantor function and consider the Borel measure μ_f, which is the Lebesgue-Stieltjes measure associated to f. It can be checked that μ_f works. □

Problem 8.4. Let (f_n) be a sequence of continuous functions on $[0, 1]$ and suppose that for all $x \in [0, 1]$, $f_n(x)$ is eventually nonnegative. Show that there is an open interval $I \subset [0, 1]$ such that for all n large enough, f_n is nonnegative everywhere on I.

Proof. Define $E_n = \{x : \forall m \ge n \; f_m(x) \ge 0\}$. Suppose the conclusion is not right. For all subinterval $I \subset [0, 1]$, $I \nsubseteq E_n$, which entails that E_n is an nowhere dense closed set. However, $[0, 1] = \bigcup_n E_n$, which is a contradiction. □

Problem 8.5. Let μ be a nonatomic signed measure on a measurable space (X, Ω), with $\mu(X) = 1$. Show that there is a measurable subset $E \subset X$ with $\mu(E) = 1/2$.

Proof. At first, assume μ is a positive measure. We show that there is a function $S : [0, 1] \to \Omega$ such that for all $0 \le t \le t' \le 1$, $\mu(S(t)) = t$ and $S(t) \subset S(t')$ (i.e. increasing function).

Let $K = \{S : D \to \Omega : D \subset [0, 1], S \text{ is increasing}, \forall t \in D,\ \mu(S(t)) = t\}$. Order K by $S \le S'$ if graph$(S) \subset$ graph(S'). At first, $K \ne \emptyset$ since $S : \{1\} \to \Omega$ by $S(1) = X$. Let $\{S_\alpha\}$ be a chain in K. Define $S : \bigcup D_\alpha \to \Omega$ by $S(t) = S_\alpha(t)$ if $t \in D_{S_\alpha}$. Then $S \in K$. Now, Zorn's lemma entails that there is a maximal element $S_0 \in K$. We claim that $D_{S_0} = [0, 1]$. Suppose not, let $u = \inf\{x : x \notin D_{S_0}\}$, if $u = 0$, we extend S_0 by define$S_0(0) = \emptyset$. If $u > 0$, there is a sequence $(u_n) \subset D_{S_0} \uparrow u$. Then we extend S_0 by define $S_0(u) = \bigcup S_0(u_n)$. It is compatible with the original S_0. Indeed, since S_0 is increasing, $\mu(S_0(u)) = \mu(\bigcup S_0(u_n)) = \lim_n u_n = u$. A contradiction to the maximality of S_0. Then $S(1/2)$ is the set we want.

Now, if μ is a signed measure. Consider a Hahn decomposition $X = P \bigcup N$. μ is positive on P with $\mu(P) \geq 1$. By the argument above, there is a E such that $\mu(E) = 1/2$. □

Problem 8.6. Compute $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)} dx$

Proof. At first, $\sin(x/n) < x/n$ implies that $\frac{n\sin(x/n)}{x(1+x^2)} \leq \frac{1}{1+x^2}$, which is an integrable function on $\mathbb{R}^+$. Then since $\frac{n\sin(x/n)}{x(1+x^2)} \to \frac{1}{1+x^2}$, DCT implies that $\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)} dx = \int_0^\infty \frac{1}{1+x^2} dx =$some number. □

Problem 8.7. Prove or disprove: for every real-valued continuous function f on $[0,1]$ such that $f(0) = 0$ and every $\epsilon > 0$, there is a real polynomial p having only odd powers of x, i.e. $p = \sum_{i=1}^n a_{2i+1}x^{2i+1}$ such that $\sup_{x\in[0,1]} |f(x) - p(x)| < \epsilon$.

Proof. At first, consider $A = \{xf(x) : f \in C[0,1]\}$ is a subalgebra which separates points by $g(x) = x$. It implies that $\overline{A} = \{f \in C[0,1] : f(0) = 0\}$ by Stone-Weirstrass theorem. Then similar argument shows that $B = \{p : p(x) = \sum_0^n a_i x^{2i}\}$ is dense in $C[0,1]$ under $\|\cdot\|_\infty$. Then for every $f \in C[0,1]$ with $f(0) = 0$, there is an element $g \in C[0,1]$ such that $\sup_{x\in[0,1]} |f(x) - xg(x)| < \epsilon/2$ since $xg(x) \in A$. For g, there is a $p \in B$ such that $\|g - p\|_\infty \leq \epsilon$ and thus $\sup_{x\in[0,1]} |xg(x) - xp(x)| < \epsilon/2$. Combine them, we have $\sup_{x\in[0,1]} |f(x) - xp(x)| < \epsilon$, where $xp(x)$ is a polynomial having only odd powers of x. □

Problem 8.8.

Problem 8.9. Let X be a separable Banach space, let $\{x_n : n \geq 1\}$ be a countable, dense subset of the unit ball of X and let B be the closed unit ball in the dual Banach space X^* of X. For $\phi, \psi \in B$, let $d(\phi, \psi) = \sum_{n=1}^\infty 2^{-n}|\phi(x_n) - \psi(x_n)|$. Show that d is a metric on B whose topology agrees with the weak*-topology of X^* restricted to B.

Proof. It suffices to show open sets in two topologies coincide. Let $P_{x^*,\epsilon,x} = \{y^* : |y^*(x) - x^*(x)| < \epsilon\}$ and $D = \{x_n\}$ is dense subset of X. if $x^* \in \bigcap_{i=1}^m P_{x_i^*,\epsilon,z_i}$. At first, By the density of D, there is a δ and $\{x_{n_k} : k = 1, 2, ..., m\}$ so that $x^* \in \bigcap_{i=1}^m P_{x_i^*,\delta,x_{n_i}} \subset \bigcap_{i=1}^m P_{x_i^*,\epsilon,z_i}$. Then, there is a r so that $d(x^*, y^*) < r$ implies that $|x^*(x_{n_k}) - y^*(x_{n_k})| < \delta$ for $k = 1, 2, ..., m$. This implies that $x^* \in B_d(x^*, r) \subset \bigcap_{i=1}^m P_{x_i^*,\delta,x_{n_i}} \subset \bigcap_{i=1}^m P_{x_i^*,\epsilon,x_i}$.

In the converse, for all $\phi, \psi \in B$, $|\phi(x_n) - \psi(x_n)| \leq \|\phi\| + \|\psi\| \leq 2$, and thus $d(\phi, \psi) = \sum_{n=1}^\infty 2^{-n}|\phi(x_n) - \psi(x_n)| \leq \sum_{n=1}^\infty 2^{-n+1}$, which implies that for every

$\epsilon > 0$, there is a $m \in \mathbb{N}$ such that for all $\phi, \psi \in B$, $\sum_{n=m+1}^{\infty} 2^{-n}|\phi(x_n) - \psi(x_n)| < \epsilon/2$. Now, for all $y^* \in B_d(x^*, \epsilon)$, $\sum_{n=m+1}^{\infty} 2^{-n}|x^*(x_n) - y^*(x_n)| < \epsilon/2$. Then $x^* \in \bigcap_{k=1}^{m} P_{x^*, \frac{\epsilon}{2m}, x_k} \subset B_d(x^*, \epsilon)$. Then we are done. □

Problem 8.10. Let $T : X \to Y$ be a linear map between Banach spaces that is surjective and satisfies $\|Tx\| \geq \epsilon\|x\|$ for some $\epsilon > 0$ and all $x \in X$. Show that T is bounded.

Proof. Let $x_n \to x$ and $Tx_n \to y$. Since T is surjective, there is a $z \in X$ such that $Tz = y$. Then $\|Tx_n - Tz\| \geq \epsilon\|x_n - z\|$ entails that $x_n \to z$ and thus $x = z$ and $Tx = y$. By closed graph theorem, T is bounded. □

9. January 2013

Problem 9.1. This problem is same with Problem 4 in Jan 2014.

Problem 9.2. This problem is same with Problem 3 in Aug 2015.

Problem 9.3. This problem is same with Problem 8 in Jan 2016.

Problem 9.4. (a) Is there a signed Borel measure μ on $[0, 1]$ such that $p'(0) = \int_0^1 p(x)d\mu(x)$ for all real polynomials p of degree at most 19.

(b) Is there a signed Borel measure μ on $[0, 1]$ such that $p'(0) = \int_0^1 p(x)d\mu(x)$ for all real polynomials p.

Proof. (a) Let $X = \text{span}\{x^i : 0 \leq i \leq 19\} \subset C[0, 1]$ is a finite-dimensional subspace. Then define $F : X \to \mathbb{R}$ by $F(p) = p'(0)$ is a linear map and thus continuous since $\dim(X) = 20$. Then by Hahn-Banach theorem, we can extend F to some bounded linear map G on $C[0, 1]$. Then there is a Radon Borel measure such that $G(f) = \int_0^1 f d\mu$. Restrict G to X, we have $p'(0) = \int_0^1 p(x)d\mu(x)$.

(b) Suppose it is true. Consider $f(x) = ax$, where $a > 0$. Then by Stone-Weirstrass theorem, there is a sequence of polynomials $p_n(x) = \sum_{i=1}^{m_n} a_{(i,n)}x^{2i} \to f$ under $\|\cdot\|_\infty$. Then $|\int_0^1 p_n d\mu - \int_0^1 f d\mu| \leq \|p_n - f\|_\infty \to$. However, by the assumption, $p_n'(0) = 0$ and thus $a = \int_0^1 d\mu = 0$. A contradiction. □

Problem 9.5. Let $\mathcal{F}$ be the set of all real valued functions on $[0, 1]$ of the form $f(t) = \frac{1}{\prod_{j=1}^{n}(t - c_j)}$ for natural numbers n and for real numbers $c_j \notin [0, 1]$. Prove or disprove: for all continuous real-valued functions g and h on $[0, 1]$ such that $g(t) < h(t)$ for all $t \in [0, 1]$, there is a function $a \in \text{span}\,\mathcal{F}$ such that $g(t) < a(t) < h(t)$ for all $t \in [0, 1]$.

Proof. It can be seen that $\mathcal{A} = \text{span}\,\mathcal{F}$ is an algebra and separates points. Then by Stone-Weirstrass theorem $\overline{\mathcal{A}} = C[0, 1]$. Let $k = \min\{h(t) - g(t)\}$. Now, choose $a_1 \in$

$\mathcal{A}$ such that $\|a_1 - g\|_\infty < k/3$, whence $g(t) < a_1(t) + k/3$ and also $a_1(t) < g(t) + k/3$. Then we need to find a a_2 such that $k/3 \le a_2(t) \le 2k/3$. Indeed, for $u \equiv k/2$, there is an $a_2 \in \mathcal{A}$ so that $\|a_2 - u\|_\infty < \epsilon < k/6$, whence $k/3 \le a_2(t) \le 2k/3$. Now, let $a = a_1 + a_2$. Then $g(t) < a(t) < g(t) + k/3 + 2k/3 \le h(t)$ □

Problem 9.6. Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous and let $1 < p < \infty$. For $f \in L^p[0,1]$, let Tf be the function on $[0,1]$ defined by $(Tf)(x) = \int_0^1 k(x,y)f(y)dy$. Show that Tf is a continuous function on $[0,1]$ and that the image under T of the unit ball in $L^p[0,1]$ has compact closure in $C[0,1]$.

Proof. $|(Tf)(x) - (Tf)(x')| \le \int_0^1 |k(x,y) - k(x',y)||f(y)|dy$. Since k is uniformly continuous, if $|x' - x| < \delta$, then for all y, $|k(x,y) - k(x',y)| < \epsilon$. Then $|(Tf)(x) - (Tf)(x')| \le \epsilon \|f\|_1$. Then Tf is continuous. It entails that $\mathcal{F} = \{T(f) : \|f\|_p \le 1\}$ is equicontinuous by $|(Tf)(x) - (Tf)(x')| \le \epsilon\|f\|_1 \le \epsilon\|f\|_p \le \epsilon$ whenever $|x' - x| < \delta$. Now, since $|k(x,y)| \le M$ on $[0,1]^2$. $|T(f)(x)| \le \int_0^1 |k(x,y)||f(y)| \le M\|f\|_1 \le M\|f\|_p \le M$ if $\|f\|_p \le M$. Then Ascoli-Arzela theorem applies there. □

Problem 9.7. Suppose $f : [0,1] \to \mathbb{R}$ is absolutely continuous and define $g \in C[0,1]$ by $g(x) = \int_0^1 f(xy)dy$. Show that g is absolutely continuous.

Proof. For all ϵ, there is a δ, such that $\sum_{i=1}^n |a_i - b_i| < \delta$ implies $\sum_{i=1}^n |f(a_i) - f(b_i)| < \epsilon$. For all $y \in [0,1]$, $\sum_{i=1}^n |a_i y - b_i y| \le \sum_{i=1}^n |a_i - b_i|\delta$, then $\sum_{i=1}^n |f(a_i y) - f(b_i y)| < \epsilon$, which implies that $\sum_{i=1}^n |g(a_i) - g(b_i)| \le \sum_{i=1}^n \int_0^1 |f(a_i y) - f(b_i y)|dy = \int_0^1 \sum_{i=1}^n |f(a_i y) - f(b_i y)|dy \le \int_0^1 \epsilon dy = \epsilon$. □

Problem 9.8. Suppose that we have $\nu_i \ll \mu_i$ for $i = 1,2$. Show that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$ and $\frac{d(\nu_1 \times \nu_1)}{d(\mu_1 \times \mu_2)}(x,y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y)$

Proof. $\mu_1 \times \mu_2(E) = \int \mu_2(E_x)d\mu_1(x) = 0$ implies that $\mu_1\{x : \mu_2(E_x) \ne 0\} = 0$. Then since $\nu_i \ll \mu_i$ for $i = 1,2$, firstly $\{x : \nu_2(E_x) \ne 0\} \subset \{x : \mu_2(E_x) \ne 0\}$ and then $\mu_1\{x : \nu_2(E_x) \ne 0\} = 0$ implies $\nu_1\{x : \nu_2(E_x) \ne 0\} = 0$. Thus $\nu_1 \times \nu_2(E) = \int \nu_2(E_x)d\nu_1(x) = 0$ which implies that $\nu_1 \times \nu_2 \ll \mu_1 \times \mu_2$. In addition, $\nu_1 \times \nu_2(E) = \int_{X \times Y} \chi_E(x,y)d(\nu_1 \times \nu_2) = \int_X (\int_Y \chi_E(x,y)f_2(y)d\mu_2(y))f_1(x)d\mu_1(x) = \int_X (\int_Y \chi_E(x,y)f_1(x)f_2(y)d\mu_2(y))d\mu_1(x) = \int_E f_1(x)f_2(y)d(\mu_1 \times \mu_2)$. We are done. □

Problem 9.9. (a) Let E be a nonzero Banach space and show that for every $x \in E$ there is $\phi \in E^*$ such that $\|\phi\| = 1$ and $|\phi(x)| = \|x\|$.

(b) Let E and F be Banach spaces, Let $\pi : E \to F$ be a bounded linear map and let $\pi^* : F^* \to E^*$ be the induced map on dual spaces. Show that $\|\pi^*\| = \|\pi\|$.

Proof. (a) A standard application of Hahn-Banach theorem.

(b) $\pi^*(f) = f \circ \pi$. Then $\|\pi^*\| = \sup_{\|f\|=1} \|\pi^*(f)\| = \sup_{\|f\|=1, \|x\|=1} |f(\pi(x))|$. $|f(\pi(x))| \leq \|f\| \|\pi\| \|x\|$ for all $f \in F^*, x \in E$, whence $\|\pi^*\| \leq \|\pi\|$. In the converse, by part (a), for every $x \in E$, there is a $f_x \in F^*$ such that $|f_x(\pi(x))| = \|\pi(x)\|$ and $\|f_x\| = 1$. Thus, $\|\pi^*\| = \sup_{\|f\|=1, \|x\|=1} |f(\pi(x))| \geq \sup_{\|x\|=1} |f_x(\pi(x))| = \|\pi\|$. Then we are done. □

Problem 9.10. Let X be a real Banach space and suppose C is a closed subset of X such that

(1) $x_1 + x_2 \in C$ for all $x_1, x_2 \in C$,

(2) $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$,

(3) for all $x \in X$ there exist $x_1, x_2 \in C$ such that $x = x_1 - x_2$.

Prove that for some $M > 0$, the unit ball of X is contained in the closure of $A_M = \{x_1 - x_2 : x_i \in C, \|x_i\| \leq M, (i = 1, 2)\}$. Deduce that every $x \in X$ can be written $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \leq 2M\|x\|$, $(i = 1, 2)$.

Proof. Suppose that there is a $\epsilon > 0, n \in \mathbb{N}$ so that $B_\epsilon(0) \subset \overline{A_n}$, then there is a m such that $B_X \subset \overline{A_m}$. Indeed, for all $x \in B_X$, $\|\epsilon x\| \leq \epsilon$, then there is a sequence $(x_1^k - x_2^k) \in A_n$ so that $(x_1^k - x_2^k) \to \epsilon x$, which implies that $\frac{1}{\epsilon}(x_1^k - x_2^k) \to x$. Now, define $y_i^k = \frac{1}{\epsilon} x_i^k$ and choose m such that $\|y_i^k\| \leq n/\epsilon \leq m$.

So suppose the assumption does not hold, say, for all ϵ, n, $B_\epsilon(0) \nsubseteq \overline{A_n}$, then $\overline{A_n}$ is nowhere dense for all n. If not, say there is a $y + B_\epsilon(0) \subset \overline{A_n}$. Then for all $x \in B_\epsilon(0)$, there is a sequence $(z_1^n - z_2^n) \to y + x$. For a sequence $(y_1^n - y_2^n) \to y$, $((z_1^n + y_2^n) - (z_2^n + y_1^n)) \to x$, which implies that $x \in \overline{A_{2n}}$ and thus $B_\epsilon(0) \subset \overline{A_{2n}}$. A contradiction. However, $X = \bigcup_{n=1}^\infty A_n$, which is a contradiction to Baire category theorem. Thus, there is a M such that $B_X \subset \overline{A_M}$.

Now, fix $x \in X$, $x/\|x\| \in B_X \subset \overline{A_M}$, then there are y_1, z_1 such that $\|y_1\|, \|z_1\| \leq M\|x\|$, so that $\|x - (y_1 - z_1)\| \leq (1/2)\|x\|$. Then $\|\frac{x - (y_1 - z_1)}{(1/2)\|x\|}\| \leq 1$. Then there are $y_2, z_2 \in A_M$..... We can do this process by induction to obtain y_n, z_n with $\|z_n\|, \|y_n\| \leq 2^{-n+1} M\|x\|$ and $\|x - \sum_{i=1}^n (y_i - z_i)\| \leq 2^{-n}\|x\|$, which implies that $x = \sum_i y_i - \sum_i z_i$. □

10. August 2012

Problem 10.1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that $L_1(X, \mu)$ is complete.

Proof. You can find a proof in any real analysis textbook. □

Problem 10.2. Fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with $\mu(X), \nu(Y) > 0$. Let $f : X \to \mathbb{C}$, and $g : X \to \mathbb{C}$ be measurable. Suppose $f(x) = g(y)$ $(\mu \times \nu)$-a.e. Show that there is a constant $a \in \mathbb{C}$ such that $f(x) = a$ μ-a.e. and $g(y) = a$ ν-a.e.

Proof. $E = \{(x,y) : f(x) \neq g(y)\}$ is null. Then $\int \nu(E_x)d\mu(x) = 0$, which implies that $\nu(E_x) = 0$ μ-a.e. Now, define $K = \{x : \nu(E_x) = 0\}$. Suppose the statement is not true, say, for all $a \in \mathbb{C}$, either $\mu\{x : f(x) \neq a\} > 0$ or $\nu\{y : g(y) \neq a\} > 0$. Now, fix $x_0 \in K$ and let $a = f(x_0)$. Then, by definition, $E_{x_0} = \{y : g(y) \neq a\}$ and thus $\nu(\{y : g(y) \neq a\}) = 0$. Thus, $\mu(\{x : f(x) \neq a\}) > 0$ by the assumption. Since $\nu(E_{x_0}) = 0$, $\nu(\{y : g(y) = a\}) = \nu(Y \setminus E_{x_0}) > 0$. Now, $\{x : f(x) \neq a\} \times \{y : g(y) = a\} \subset E$ is of positive measure. A contradiction. □

Problem 10.3. Let $\mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable function. Suppose for every ball B, f is integrable on B and $\int_B f = 0$. What can you deduce about f.

Proof. Since f is locally integrable, then $\lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy = f(x)$ for a.e. x, which implies that $f(x) = 0$ a.e.

We can also use the proof of Problem 1 in Jan 2015 to solve this. Indeed, we can still define E_n. By regularity of lebesgue measure. We can shrink E_n to some compact subset K of positive measure s and obtain a finite cover of K with finite balls, say $K \subset O = \bigcup_{i=1}^n B_n$ with $\mu(O \setminus K) < \delta$. Then replace E_n with K, I_n with B_n to do the same process. We are done. □

Problem 10.4. Let X be a locally compact Hausdorff space. Show that $C_c(X)$ is dense in $C_0(X)$.

Proof. $C_c(X)$ is an algebra. by Usysohn's lemma, for all $x \neq y \in X$, there is a $g \in C_c(X)$ so that $g(x) \neq g(y)$ and for all $x \in X$, there is a function f in $C_c(X)$ such that $f(x) = 1$. Then by Stone-Weirstrass theorem for the LCH space, $C_c(X)$ is dense in $C_0(X)$. □

Problem 10.5. Give an example of each of the following. justify your answers

(1) A nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure.

(2) A closed, convex subset of a Banach space with multiple points of minimal norm.

Proof. (1) A first example is any generalized cantor set with positive measure. We provide another example with more descriptive set theory flavor. Let $\{r_n\}$ enumerate all rational numbers in $\mathbb{R}$, then define $O_m = \bigcup_{n=1}^\infty (r_n - \frac{1}{2^{n+1}}\frac{1}{m}, r_n + \frac{1}{2^{n+1}}\frac{1}{m})$ which is dense and $\mu(O_m) \leq 1/m$. Then $A = \bigcap_{m=1}^\infty O_m$ is a dense G_δ set with $\mu(A) = 0$. Then $A^c = \bigcup_m O_m^c$ is of positive measure. Then there is a m such that O_m^c is of positive measure and O_m^c is nowhere dense by definition.

(2) Let $X = L^1[0,1]$ and $C = \{f \in X : \int_{[0,1]} f(t)dt = 1\}$. The minimal norm of functions in C is 1. Then $\{a\chi_{[1/2,1]} + (2-a)\chi_{[0,1/2]}\} : 0 \geq a \geq 2\} \subset C$ have norm 1. □

Problem 10.6. Let $S = \{f \in L^\infty(\mathbb{R}) : |f(x)| \le \frac{1}{1+x^2} a.e.\}$. Which of the following statements are true?

(1) The closure of S is compact in the norm topology.

(2) S is closed in the norm topology.

(3) The closure of S is compact in the weak-$*$ topology.

Proof. (1) No. Consider a sequence $\frac{1}{1+nx^2}$. If $\overline{S}$ is compact, then there is a subsequence $\frac{1}{1+n_m x^2}$ converges in $\overline{S}$ under $\|\cdot\|_\infty$. Then $\frac{1}{1+n_m x^2} \to 0$ uniformly a.e. , which is a contradiction since $\frac{1}{1+n_m x^2}$ does not uniformly converge to 0 on any set of infinite measure.

(2) For all $f_n \to f$ under $\|\cdot\|_\infty$. Define E_n be the null set such that on E_n^c, $|f_n(x)| \le \frac{1}{1+x^2}$. Since $f_n \to f$, there is a null set E such on E^c, $f_n \to f$ uniformly. Then on $(\bigcup_n E_n \bigcup E)^c$, $|f(x)| \le \frac{1}{1+x^2}$, where $\mu((\bigcup_n E_n \bigcup E)) = 0$

(3) For all $f \in S$, $|f(x)| \le \frac{1}{1+x^2}$ a.e. which implies that $\|f\|_\infty \le 1$. Thus $S \subset B_{X^*}$. Thus S is weak-$*$ compact. □

Problem 10.7. Let T be a bounded operator on a Hilbert space H. Prove that $\|T^*T\| = \|T\|^2$.

Proof. $\|T\|^2 = \sup_{\|x\|=1} \|Tx\|^2 = \sup_{\|x\|=1} |\langle Tx, Tx\rangle| = \sup_{\|x\|=1} |\langle x, T^*Tx\rangle| \le \sup_{\|x\|=1} \|x\|\|T^*Tx\| \le \|T^*T\|$. In the converse, $\|T^*Tx\| \le \|T^*\|\|T\|\|x\| = \|T\|^2$ implies that $\|T^*T\| \le \|T\|^2$. □

Problem 10.8. (a) Let g be an integrable function on $[0, 1]$. Does there exist a bounded measurable function f such that $\|f\|_\infty \ne 0$ and $\int_0^1 fgdx = \|g\|_1\|f\|_\infty$.

(b) Let g be a bounded function on $[0, 1]$. Does there exist an integrable measurable function f such that $\|f\|_1 \ne 0$ and $\int_0^1 fgdx = \|f\|_1\|g\|_\infty$.

Proof. (a) Define $f = \overline{\mathrm{sgn} g}$. Then $\int_0^1 fgdx = \|g\|_1\|f\|_\infty$.

(b) Not always. Consider $g = 1_{\{x\}}$ for some $x \in [0, 1]$. Suppose there is a $f \in L_1[0, 1]$ so that $\int_0^1 fgdx = \|f\|_1\|g\|_\infty$. Then it implies that $\|f\|_1 = \int_{\{x\}} f = 0$. □

Problem 10.9. Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function, μ the Lebesgue measure, and $f, g \in L^1(\mu)$. Let $\tilde{f}(x) = \int F(xy)f(y)d\mu(y)$, $\tilde{g}(x) = \int F(xy)g(y)d\mu(y)$. Show that $\tilde{f}$ and $\tilde{g}$ are bounded continuous functions which satisfy $\int f\tilde{g}d\mu = \int g\tilde{f}d\mu$

Proof. Let $\sup_{x\in\mathbb{R}} |F(x)| = M$. Consider $|\tilde{f}(x_n) - \tilde{f}(x)| \le \int |F(x_n y) - F(xy)||f(y)|dy$. Since $|F(x_n y) - F(xy)||f(y)| \to 0$ pointwise and $|F(x_n y) - F(xy)||f(y)| \le 2M|f(y)|$, which is integrable, DCT implies that $\int |F(x_n y) - F(xy)||f(y)|dy = 0$. Thus $\tilde{f}$ is continuous. $|\tilde{f}(x)| \le \int |F(xy)||f(y)|d\mu(y) \le M\|f\|_1$, which is bounded. The same proof for $\tilde{g}$.

$\int |f(x)|(\int |F(xy)||g(y)|d\mu(y))d\mu(x) \leq M \int |f(x)|\|g\|_1 d\mu(x) \leq M\|f\|_1\|g\|_1$. Then, by Fubini, $\int f\tilde{g}d\mu = \int f(x)(\int F(xy)g(y)d\mu(y))d\mu(x) = \int g(y)(\int F(xy)f(x)d\mu(x))d\mu(y) = \int g\tilde{f}d\mu$. □

Problem 10.10. Let $\mu, \{\mu_n : n \in \mathbb{N}\}$ be finite Borel measures on $[0,1]$. $\mu_n \to \mu$ vaguely if $\mu_n \to \mu$ if it converges in the weak*-topology. $\mu_n \to \mu$ in moments if for each $k \in \{0\} \bigcup \mathbb{N}$, $\int_{[0,1]} x^k d\mu_n \to \int_{[0,1]} x^k d\mu$. Show these two concepts coincide.

Proof. ($\Rightarrow$) trivial, since $x^k \in C[0,1]$.

($\Leftarrow$) For $k = 0$, we have $\mu_n([0,1]) \to \mu([0,1])$, which implies that $\mu_n([0,1])$ are uniformly bounded, say, we may assume that $\sup\{\mu_n[0,1], \mu[0,1] : n \in \mathbb{N}\} \leq M$ for some M. By Stone-Weirstrass theorem, for each continuous function f on $[0,1]$, there is a sequence of polynomials $p_m \to f$ under $\|\cdot\|_\infty$. Then for every $\epsilon > 0$, choose a m such that $\|p_m - f\|_\infty \leq \epsilon/4M$. For the m, there is a N whenever $n > N$ such that $|\int p_m d\mu_n - p_m d\mu| \leq \epsilon/2$ since $\mu_n \to \mu$ in moments. Then $|\int f d\mu_n - \int f d\mu| \leq \int |f - p_m|d\mu_n + \int |f - p_m|d\mu + |\int p_m d\mu_n - p_m d\mu| \leq 2M \cdot (\epsilon/4M) + \epsilon/2 = \epsilon$. Thus $\mu_n \to \mu$ vaguely. □

11. January 2012

Problem 11.1. Let A be the subset of $[0,1]$ consisting of numbers whose decimal expansions contain no sevens. Show that A is Lebesgue measurable, and find its measure. Why does non-uniqueness of decimal expansions not cause any problems?

Proof. We can obtain A by following the procedure of the construction of the cantor set. Firstly divide $[0,1]$ into ten pieces and delete $(0.7, 0.8]$ to obtain A_1. Then to obtain A_2, for each of the nine subintervals of A_1, divide it into ten pieces and delete the seventh subinterval of it. Define the left set to be A_2.Go on this procedure to obtain A_n. Then define $A = \bigcap_n A_n$. Since we just delete Borel sets from $[0,1]$. A is Borel thus measurable. $\mu(A) = 1 - \sum_{n=0}^{\infty} \frac{9^n}{10^{n+1}} = 0$.

Since the number who has more than one expansion are exactly $\{\frac{p}{10^n} : n, p \in \mathbb{N}, p \leq 9\}$ which is countable, thus of measure 0. So it will cause no problems. □

Problem 11.2. Let functions f_α be defined by $f_\alpha(x) = x^\alpha cos(1/x)$ if $x > 0$ and $f_\alpha(0) = 0$. Find all $\alpha \geq 0$ such that

(a) f_α is continuous.

(b) f_α is of bounded variation on $[0,1]$.

(C) f_α is absolutely continuous.

Proof. $|f_\alpha(x)| = | = x^\alpha \cos(1/x)| \leq |x^\alpha| \to 0$ when $x \to 0$ if $\alpha > 0$. Thus for all $\alpha > 0$, f_α is continuous. For part (b), if $\alpha \leq 1$, define $x_n = 1/n\pi$, $\sum |f_\alpha(x_n) - f_\alpha(x_{n+1}|$

diverges, which implies that f_α is not of bounded variation. In the converse, at first, f_α is differentiable on $(0,1]$ for all $\alpha > 0$. If $\alpha > 1$, f_α is also differentiable at 0. Then $f_\alpha(x) = \int_0^x f'_\alpha(t)dt$. Thus, f_α is of bounded variation. For part (c), If $\alpha \leq 1$, f_α is not of bounded variation, whence f_α is not absolutely continuous. When $\alpha > 1$, f is of a form of integral, thus absolutely continuous. □

Problem 11.3. Let $\mathcal{F}$ denote the family of functions on $[0,1]$ of the form $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$, where a_n are real and $|a_n| \leq 1/n^3$. Prove that $\mathcal{F}$ is precompact i.e. totally bounded.

Proof. For all $x, y \in [0,1]$, there is a x_0 between them and a $M > 0$ such that$|f(x) - f(y)| \leq \sum |a_n||\sin(nx) - \sin(ny)| \leq \sum |a_n||\cos(nx_0)||nx - ny| \leq \sum_n \frac{1}{n^2}|x-y| \leq M|x-y|$, whence $\mathcal{F}$ is equicontinuous on $[0,1]$. For all $x \in [0,1]$, $\{f(x) : f \in \mathcal{F}\}$ is bounded since $|f(x)| \leq \sum_n 1/n^3 \leq \infty$. Then Ascoli-Arzela's theorem implies the result. □

Problem 11.4. Let H be a Hilbert space and $W \subset H$ be a subspace. Show that $H = \overline{W} \bigoplus W^{\perp}$

Proof. You can find the proof in every textbook of functional analysis. □

Problem 11.5. Suppose A is a bounded linear operator on a Hilbert space H with the property that $\|p(A)\| \leq C \sup\{|p(z)| : z \in S^1\} = C\|p\|$ with complex coefficients and a fixed constant C. Show that to each pair $x, y \in H$, there corresponds a complex Borel measure μ on the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ such that $\langle A^n x, y\rangle = \int z^n d\mu(z)$, $n = 0, 1, 2, ...$

Proof. Define $H(p) = \langle p(A)x, y\rangle$ for all polynomials p with complex coefficients. $|H(p)| = |\langle p(A)x, y\rangle| \leq \|p(A)\|\|x\|\|y\| \leq C\|p\|\|x\|\|y\|$, which implies that H is extended to whole $C(S^1)$ by the density of all polynomials in $C(S^1)$ as a bounded linear functional. Thus, there is a Radon measure μ on S^1 such that $H(f) = \int f d\mu$. Then $H(z^n) = \langle A^n x, y\rangle = \int z^n d\mu(z)$, $n = 0, 1, 2, ...$ □

Problem 11.6. Let ϕ be the linear functional $\phi(f) = f(0) - \int_{-1}^{1} f(t)dt$.

(a) Compute the norm of ϕ as a functional on the Banach space $C[-1,1]$ with uniform norm.

(b) Compute the norm of ϕ as a functional on the normed vector space $LC[-1,1]$, which is $C[-1,1]$ with L^1 norm.

Proof. (a) $\|\phi\| = \sup\{|\phi(f)| : \|f\|_\infty = 1\}$. $\|f\|_\infty = 1$ implies that $|f(0) - \int_{-1}^{1} f(t)dt| \leq |f(0)| + \int_{-1}^{1} |f(t)|dt = 3$. Consider the continuous function f_n such that $f_n = 1$ on $[-1, -1/n] \bigcup [1/n, 1]$, $f_n(0) = -1$ and f_n is peicewise linear on $[-1/n, 1/n]$. Then, we can see $|\phi(f_n)| \to 3$, whence $\|\phi\| = 3$.

(b) Define $f_n = 0$ on $[-1, -1/n] \bigcup [1/n, 1]$, $f_n(0) = n$ and f_n is peicewise linear on $[-1/n, 1/n]$. $\|f_n\|_1 = 1$ and $|\phi(f_n)| = n + \int_{-1}^{1} |f_n| = n + 1$ which implies that $\|\phi\| = \infty$. □

Problem 11.7. Let X be a normed space, and $A \subset X$ a subset. Show that A is bounded(as a set) if and only if it is weakly bounded(that is, $f(A) \subset \mathbb{C}$ is bounded for each $f \in X^*$).

Proof. ($\Rightarrow$) Suppose that $\sup\{\|x\| : x \in A\} \leq M$. Then for all $f \in X^*$, $|f(x)| \leq \|f\|\|x\| \leq M\|f\|$.

($\Leftarrow$) X^* is a Banach space. For all $f \in X^*$, $x^{**}(f) = f(x)$. Now $\sup_{x \in A} |x^{**}(f)| = \sup_{x \in A} |f(x)| < \infty$. Then uniform boundedness theorem implies that $\sup_{x \in A} \|x\| = \sup_{x \in A} \|x^{**}\| < \infty$. □

Problem 11.8.

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