# Math 871 - Table of contents

## Chapter 1: Preliminaries

- *Section A: Overview of the course*
  - Spaces we live in - Euclidean room, spherical world, hyperbolic universe
  - Describe these spaces without "distance"
  - When does calculus, and in particular the Extreme Value Theorem, apply to these spaces?
- *Section B: Useful tools on sets and functions*
  - De Morgan's laws
  - Thm: Preimages preserve all Boolean constructions, images preserve some.
    "Preimages are nice"
  - Tips on how to prove sets are finite or countable.
  - Axiom of choice and arbitrary Cartesian products

## Chapter 2: Topological spaces and continuous functions

- *Section A: Topology and continuity*
  - Topology:
    - Open sets in Euclidean space
    - Definition of a topology
    - Examples: Discrete, indiscrete, Euclidean, $S^2$, finite complement, included point, excluded point, infinite ray, line with 2 origins, finite
    - Finer and coarser topologies on the same set
  - Continuity:
    - Motivation: Unraveling the ε-δ definition of continuous in Euclidean space to an equivalent definition on open sets.
    - Definition of continuous function
    - Examples, including inverse image topology
    - Thm: Constant functions and composition of continuous functions are continuous
    - Definition of homeomorphism
    - Examples of pairs of spaces that are or are not homeomorphic
    - Big picture:
      - Euclidean, spherical, and hyperbolic spaces are homeomorphic; topology enables proofs for all three at once
      - Big questions in topology: Homeomorphism and Classification Problems
  - Bases:
    - Definitions of basis and topology generated by a basis
    - Lemmas: (1) The topology generated by a basis is a topology. (2) *T(B)* is the collection of unions of basis elements.
    - Thm: A function f is continuous iff preimages of basis elements are open.
    - Examples, including Euclidean and lower limit topology

    - Lemma: If *T* is a topology, then *T* is a basis for the topology *T*.
    - Thm: *T(B)* ⊆ *T(B')* iff for all B in *B* and x in B, there is a B' in *B'* with x ∈ B' ⊆ B.

- *Section B: Constructing new spaces and continuous functions from old ones, part I*

  - Subspaces:
    - Definition and examples; "open in" or "open relative to"
    - Thm: A subspace of a subspace is a subspace
    - Thm: Given Y ⊆ (X,*T(B)*), the subspace topology on Y has a basis $B_{\text{subsp}}$ = { B ∩ Y | B ∈ B }.
    - Thms: Let X and Y be topological spaces, let A be a subspace of X, and let B be a subspace of Y.
      - The inclusion i: A → X is continuous.
      - If f: X → Y is continuous, then $f|_A$: A → Y is continuous.
      - If f: X → Y is continuous and f(X) ⊆ B, then $f|^B$: X → B is continuous.
      - If f: X → B is continuous then $f|^Y$: X → Y is continuous.

      "Restrictions and extensions of continuous functions are continuous."
    - Thm: If X = $\cup_\alpha U_\alpha$ with each $U_\alpha$ open in X and if f: X → Y is a function satisfying $f|_{U_\alpha}: U_\alpha \to Y$ is continuous for all α, then f is continuous.
    - Examples of homeomorphic Euclidean subspaces
  - Product spaces:
    - Cartesian product and projection maps
    - Definition of product and box topologies, and examples
    - Thm: A product of subspaces is a subspace of the product.
    - Definitions of subbasis, and basis and topology generated by a subbasis
    - Thm: A function f is continuous iff preimages of subbasis elements are open.
    - For an arbitrary collection of spaces $(X_\alpha, T_\alpha)$ the product topology on $\prod X_\alpha$ has a subbasis $S_{\text{prod}} = \{ p_\alpha^{-1}(U_\alpha) \mid U_\alpha \in T_\alpha \}$ and a basis $B_{\text{prod}} = \{ \prod B_\alpha \mid B_\alpha \in T_\alpha, B_\alpha = X_\alpha$ for all but finitely many α }.
    - Given topological spaces $(X_\alpha, T(B_\alpha))$ generated by bases, the product topology on $\prod X_\alpha$ has a basis $B'_{\text{prod}} = \{ \prod B_\alpha \mid B_\alpha \in B_\alpha \cup \{X_\alpha\}, B_\alpha = X_\alpha$ for all but finitely many α }.
    - The projection map $\pi_\beta : \prod X_\alpha \to X_\beta$ is continuous.
    - The function $f = (f_\alpha)_{\alpha \in J}: A \to \prod X_\alpha$ is continuous iff each $f_\alpha: A \to X_\alpha$ is continuous, using the product topology.

- *Section C: Closed sets, boundaries, and continuity*

  - Closed, definition and examples
  - Lemma: ∅, X, and finite unions and arbitrary intersections of closed sets are closed in X.
  - Prop: A closed in Y iff A = C &cap Y for some C closed in X.
  - Closure, interior, boundary, limit point: Definitions and examples
  - Prop.: C is closed iff C = $Cl_X(C)$ (ie the closure of C in X). U is open iff U = $Int_X$ U (ie the interior of U in X).
  - Thm: Closure in subspace equals intersection of closure with subspace.

    - Thm: The product of the closures equals the closure of the product.
    - Thm: TFAE: f: X &rarr Y is continuous; preimages of closed sets are closed; images of closures are contained in closures of images.
    - Thm: $Cl(A)=A \cup A'$ (where A' is the set of limit points).
    - Thm: $x \in Cl(A)$ iff every open U in X that contains x intersects A.

- *Section D: Brief review of continuity*

    - Thm: TFAE: f: $X \to Y$ is continuous; preimages of closed sets are closed; images of closures are contained in closures of images; and for all $V \in T_Y$ and $f(x) \in V$, there exist $U \in T_X$ with $x \in U$ and $f(U) \subseteq V$.
    - Thm CC: Constructing continuous functions: (1) Constant function, (2) inclusion, (3) restricting domain, (4) restricting or (5) extending range, (6) composition, (7) projection, (8) product, pasting over (9) open or (10) closed sets, (11) operations in Euclidean reals, (12) uniform limit (cosine and sine).

- *Section E: Constructing new spaces and continuous functions from old ones, part II*

    - Quotient = identification spaces:
        - Definitions of quotient topology, quotient map, and identification topology
        - Examples
        - Thm I: If p: X -> X/~ is a quotient map and g: X -> Z is continuous and constant on equivalence classes, then g induces a continuous f: X/~ -> Z with f o p = g. Moreover: (a) If [ g(x)=g(x') implies p(x)=p(x') ] then f is one-to-one; (b) If g is onto then f is onto; (c) If (a) holds and g is a quotient map then f is a homeomorphism.
        - Thm Q: If g: $X \to Z$ is continuous, onto, and open, then g is a quotient map.
        - Prop CO: Composition, projections, products, and restrictions (range) of open functions are open.
        - 2-dimensional (surface) examples

**Chapter 3: Homeomorphism invariants**

- *Section A: Motivation and Hausdorff*

    - Definition of homeomorphism invariant; example (finite)
    - Idea: Use homeomorphism invariants to prove two spaces are NOT homeomorphic.
    - Hausdorff, definition and many examples
    - Prop: Hausdorff is a homeomorphism invariant.
    - Prop: Hausdorff is preserved by subspaces and products but not quotients, continuous images, or continuous preimages.

- *Section B: Metrizability*

    - Metrizable, definition and examples
    - Prop: Metrizability is a homeomorphism invariant
    - Prop: Metrizability is preserved by subspaces and finite and countable products but not quotients, continuous images, or continuous preimages.

- Thm: Metrizable spaces are Hausdorff.

- *Section C: Connectedness*

  - Connected:
    - Motivation: Characterize spaces for which the Intermediate Value Theorem holds.
    - Definition of connected and examples
    - Thm (IVT): Let X be connected and let f: X → **R** be continuous. If there are p,q ∈ X and r ∈ **R** with f(p) < r < f(q), then there is an x ∈ X with f(x)=r.
    - Thm: A continuous image of a connected space is connected.
    - Cor: Connectedness is a homeomorphism invariant.
    - Thm: X is connected iff the only clopen subsets are ∅ and X.
    - Thm: If $X = \cup X_\alpha$, each $X_\alpha$ is connected, and $\cap X_\alpha \neq \emptyset$, then X is connected.
    - Prop: If Y ⊆ X is connected and A, B disconnect X, then either Y ⊆ A or Y ⊆ B.
    - Thm: Connectedness is preserved by continuous images, products and quotients but not subspaces or continuous preimages.
    - Thm: Y ⊆ **R** (with Euclidean top.) is connected iff Y is an interval, ray, or **R**.
    - Prop: **R** and $\mathbf{R}^n$ (with n ≥ 2) with the Euclidean topology are not homeomorphic.
  - Path connected:
    - Definition of path connected and examples
    - Thm: A continuous image of a path connected space is path connected.
    - Cor: Path connectedness is a homeomorphism invariant.
    - Thm: Path connectedness implies connectedness.
    - Prop: There are spaces that are connected but not path connected.
    - Examples (flea and comb, topologist's sine)
    - Thm: Y ⊆ **R** (with Euclidean top.) is path connected iff Y is an interval, ray, or **R**.
    - Prop: Path connectedness is preserved by products and quotients but not subspaces.
  - Components:
    - Definition of component and path component and examples
    - Prop: The number (cardinality) of (path) components is a homeomorphism invariant.
    - Prop: Each connected component is a disjoint union of path components.

- *Section D: Compactness*

  - Motivation: Characterize spaces for which the Extreme Value Theorem holds.
  - Definition of compact and examples
  - Thm (EVT): Let X be compact and let f: X → **R** be continuous. Then there are p,q ∈ X such that for all x ∈ X, f(p) ≤ f(x) ≤ f(q).
  - Thm: The continuous image of a compact space is compact.
  - Prop: Compactness is a homeomorphism invariant preserved by products and quotients but not subspaces.
  - Prop: Y ⊆ X is compact iff every covering of Y by open sets of X has finite subcovering.
  - Thm: Every closed subset of a compact space is compact.
  - Thm: Every compact subspace of a Hausdorff space is closed.
  - Prop: If X is Hausdorff, Y ⊆ X is compact, and x ∉ Y, then there are disjoint open U,V with x ∈ U, Y ⊆ V.

- Thm (Heine-Borel): Y ⊆ **R**^n is compact iff Y is closed and bounded.
- Thm (VUT): A continuous bijection from a compact space to a Hausdorff space is a homeomorphism.
- Cor (CHI): If g:X -> Y is a continuous surjection, $x_1 \sim x_2$ iff g(x1)=g($x_2$) for all $x_1$,$x_2$ ∈ X, X is compact, and Y is Hausdorff, then the quotient X/~ ≅ Y.
- Lemma (Tube): If x ∈ X, Y is compact, and N is an open set in X × Y containing {x} × Y, then there is a set W open in X with {x} × Y ⊆ W × Y ⊆ N.
- Thm (Tychonoff): The product of compact spaces is compact.
- Thm (Lebesgue Number Lemma): If X is a compact metrizable space and CC is an open covering of X, then there is a real number s > 0 such that whenever A is a subset of X with diameter < s, then there is an open set U in CC with A ⊆ U.

- *Section E: Separation properties*

  - Motivation: Characterize metrizability (in particular for compact spaces) in terms of properties defined via open sets.
  - Thm: A countable basis is a homeomorphism invariant preserved by subspaces and countable products.
  - Thm: Compact metrizable implies a countable basis, but metrizable alone does not imply a countable basis.
  - Definition of $T_0$, $T_1$, $T_2$=Hausdorff, $T_3$=regular, $T_4$=normal
  - Thm: $T_i$ for $0 \leq i \leq 4$ are homeomorphism invariant properties.
  - Lemma: $T_1$ iff all one-points sets are closed.
  - Lemma: $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. For all i, ⇐ does not hold.
  - Thm: Hausdorff and regular are preserved by subspaces and products.
  - Thm: Metrizable ⇒ normal.
  - Thm: Compact and Hausdorff ⇒ normal.
  - Thm: Regular iff $T_1$ and for all x ∈ X and open U with x ∈ U, there is an open V with x ∈ V ⊆ Cl(V) ⊆ U.
  - Thm: Normal iff $T_1$ and for all closed A and open U with A ⊆ U, there is an open V with A ⊆ V ⊆ Cl(V) ⊆ U.
  - Thm: Regular and countable basis ⇒ normal.
  - Thm (Urysohn Lemma): Let X be a $T_1$ space. TFAE: (1) X is normal. (2) Whenever A and B are disjoint closed subsets of X, there is a continuous function f : X → [0,1] such that f(A) = {0} and f(B) = {1}.
  - Thm (Urysohn Metrization): Regular and countable basis ⇒ metrizable.
  - Cor: Let X be compact. TFAE: (1) X is metrizable. (2) X is Hausdorff and has a countable basis.

**Chapter 4: Homotopy**

- *Section A: Overview of algebraic topology*

  - Big questions in topology
    - Classification problem
    - Homeomorphism problem

        - Homotopy equivalence problem
    - Algebraic homeomorphism invariants
        - Groups - homotopy groups
        - Abelian groups - homology groups
        - "Categories" and "functors"

- *Section B: Retracts*

    - Definition of retract (space) and retraction (map)
    - Definition of deformation retract (space) and deformation retraction (homotopy/map)
    - Disjoint union topology revisited
    - Mapping cylinder and mapping torus constructions; examples
    - Thm: For the mapping cylinder $X_f$ of the map f: X -> Y, Y is a deformation retract of X.
    - Examples

- *Section C: Homotopy*

    - Definitions of homotopy and homotopy relative to a subspace
    - Definitions of homotopy equivalence (maps) and homotopy equivalent/type (equivalence class of spaces)
    - Lemma: Homotopy is an equivalence relation on maps, homotopy equivalence is an equivalence relation on spaces.
    - Definition of homotopy type invariant
    - Idea: Use homotopy type invariants to prove two spaces do NOT have the same homotopy type.
    - Thm: If Y is a deformation retract of X, then they have the same homotopy type.
    - Thm: If X and Y have the same homotopy type, then there is a space Z that deformation retracts to both X and Y.
    - Cor: Let ≈ be the smallest equivalence relation on spaces such that whenever Y is a deformation retract of X then Y ≈ X. Then X ≈ Z iff X and Z are homotopy equivalent.
    - Contractible space, nullhomotopic map
    - Examples

**Chapter 5: Fundamental groups**

- *Section A: Definition of $\pi_1$*

    - Loop, path homotopy, basepoint
    - Product of loops, constant loop, reverse of a loop
    - Def: $\pi_1(X)$
    - Thm: $\pi_1$ is a group.
    - Examples: Discrete and indiscrete, $\mathbf{R}^n$ and the straight-line homotopy, $S^2$

- *Section B: Homomorphisms*

    - Change of basepoint map induced by a path
    - Thm: $\pi_1(X)$ is independent of basepoint, up to isomorphism.

- Group homomorphism induced by a map of pointed spaces
- Lemmata: Maps induced by continuous functions are group homomorphisms that compose nicely and commute with maps induced by paths.
- Thm: If X and Y are path-connected and have the same homotopy type, then $\pi_1(X) \cong \pi_1(Y)$.
- Cor: The fundamental group of a contractible space is trivial.

- *Section C: Manifolds*
  - Definitions and examples
  - Classification of compact connected 1-manifolds
  - Connected sum operation
  - Classification of compact connected 2-manifolds (surfaces)

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