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MTH U565 **TOPOLOGY** **Spring 2006**

Solutions for the Midterm Exam

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1. Let $f\colon X \to Y$ be a continuous surjection, and suppose f is a closed map. Let $g\colon Y \to Z$ be a function so that $g \circ f\colon X \to Z$ is continuous. Show that g is continuous.

Proof. It is enough to show: For every closed subset $F \subset Z$, the subset $g^{-1}(F) \subset Y$ is closed.

Now, by continuity of $g \circ f$, we know that $(g \circ f)^{-1}(F) = f^{-1}(g^{-1}(F))$ is a closed subset of X. Since f is a closed map, it takes this closed subset of X to a closed subset of Y. But

$$f((g \circ f)^{-1}(F)) = f(f^{-1}(g^{-1}(F))) = g^{-1}(F),$$

since f is surjective. Hence, $g^{-1}(F)$ is closed. □

2. Let X be a space. Show that X is Hausdorff if, and only if, the diagonal $\Delta := \{(x,x) \mid x \in X\}$ is a closed subspace of $X \times X$.

Proof. Suppose X is a Hausdorff space. We need to show that the complement of the diagonal, $\Delta^{c} := X \times X \setminus \Delta$, is open. So let $(x,y) \in \Delta^{c}$. Then $x \neq y$, and so there are disjoint open sets U and V, containing x and y, respectively. By definition of the product topology, $U \times V$ is an open subset of $X \times X$, and clearly $U \times V \subset \Delta^{c}$ (for otherwise $U \cap V \neq \emptyset$). This shows that Δ^{c} is open.

Conversely, suppose Δ is closed, that is to say, Δ^{c} is open. Let x and y be two distinct elements of X. Then $(x,y) \in \Delta^{c}$, and so there is a basis open set $U \times V \subset \Delta^{c}$ containing (x,y). Now note that U and V are open, disjoint subsets of X, containing x and y, respectively. This shows that X is Hausdorff. □

3. Let $X = [0,1]/(\frac{1}{4}, \frac{3}{4})$ be the quotient space of the unit interval, where the open interval $(\frac{1}{4}, \frac{3}{4})$ is identified to a single point. Show that X is not a Hausdorff space.

Proof. Recall that in a quotient space $X/A = (X \setminus A) \coprod \{*\}$, the open sets are of one of two types:

(1) either an open set in $X \setminus A$; or

(2) of the form $\{*\} \cup (W \cap (X \setminus A))$, where W is an open set in X, containing A.

In our situation, $X = [0,1]$ and $A = (\frac{1}{4}, \frac{3}{4})$. Take $x = \frac{1}{4}$ and $y = \frac{3}{4}$, viewed as elements of X/A. Suppose U and and V are open, disjoint subsets of X/A, containing x and y, respectively. Then, necessarily, both U and V must be of type

(2), since an open subset of $[0, 1]$ containing one of the endpoints of the interval $(\frac{1}{4}, \frac{3}{4})$ must intersect that interval. But then both U and V must contain the element $\{*\}$, and thus cannot be disjoint—a contradiction. □

4. Let X be a Hausdorff space. Suppose A is a compact subspace, and $x \in X \setminus A$. Show that there exist disjoint open sets U and V containing A and x, respectively.

Proof. Let $y \in A$. Since $x \in X \setminus A$, we see that $y \neq x$. Since X is Hausdorff, there are open, disjoint sets U_y and V_y containing y and x, respectively.

Now note that $\{U_y\}_{y\in A}$ is an open cover of A. Since A is compact, this cover admits a finite subcover, say, $U_{y_1}, \ldots, U_{y_n}$. Define:

$$U := \bigcup_{i=1}^{n} U_{y_i} \quad \text{and} \quad V := \bigcap_{i=1}^{n} V_{y_i}.$$

It is readily seen that U and V are the desired open sets. □

5. Let $p\colon X \to Y$ be a quotient map. Suppose Y is connected, and, for each $y \in Y$, the subspace $p^{-1}(\{y\})$ is connected. Show that X is connected.

Proof. Suppose X is disconnected, that is, there are disjoint, open, non-empty sets U and V such that $X = U \cup V$.

Consider the subsets $p(U)$ and $p(V)$ of Y: they are both open (since U and V are open, and p is a quotient map), and non-empty (since U and V are non-empty). Thus, by the connectivity of Y, the sets $p(U)$ and $p(V)$ cannot be disjoint.

So let $y \in p(U) \cap p(V)$. We then have

$$p^{-1}(\{y\}) = \left(U \cap p^{-1}(\{y\})\right) \cup \left(V \cap p^{-1}(\{y\})\right).$$

Both sets on the right side are open subsets of $p^{-1}(\{y\})$ (by definition of the subspace topology), and both are non-empty (since $y \in p(U)$ means $y = p(x)$, for some $x \in U$, and so $x \in U \cap p^{-1}(\{y\})$, and similarly for the other subset). Thus, by the connectivity of $p^{-1}(\{y\})$, these sets $U \cap p^{-1}(\{y\})$ and $V \cap p^{-1}(\{y\})$ cannot be disjoint. This means there is a $z \in U \cap V \cap p^{-1}(\{y\})$. Consequently, $U \cap V \neq \emptyset$, a contradiction. □

6. Let X be a discrete topological space, and let $\sim$ be an equivalence relation on X. Prove that $X/\!\sim$, endowed with the quotient topology, is also a discrete space.

Proof. Let $p\colon X \to X/\!\sim$ be the quotient map. By definition of quotient topology, a subset U of $X/\!\sim$ is open if and only if $p^{-1}(U)$ is an open subset of X. But every subset of X is open (since X has the discrete topology). Hence, every subset of $X/\!\sim$ is open; that is to say, $X/\!\sim$ is discrete. □