Title

D. Zack Garza

Sunday 17th May, 2020

Contents

1 Fall 2009

1. (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

   (2) Deduce Liouville's theorem from (1).

2. Let f be a continuous function in the region

$$D = \left\{ z \,\middle|\, |z| > R, 0 \leq \arg z \leq \theta \right\} \quad \text{where} \quad 1 \leq \theta \leq 2\pi.$$

   If there exists k such that $\lim_{z \longrightarrow \infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R' \longrightarrow \infty} \int_L f(z)dz = i\theta k,$$

   where L is the part of the circle $|z| = R'$ which lies in the region D.

3. Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where } q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z \in K} \dfrac{|z-a|}{|f(z)|}$.

(2) Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that $\dfrac{G(w)}{F'(w)} = \dfrac{1}{2\pi i} \displaystyle\int_K \dfrac{G(z)}{F(z)} dz$, where w is the zero from (1).

(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \dfrac{(qf(z))^n}{(z-a)^{n+1}}$.

4. Evaluate

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx.$$

5. Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6. Show that $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

7. For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^\infty e^{-t} t^{s-1} dt$.

   1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
   2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \displaystyle\int_0^\infty e^{-vt}(vt)^{-s} dv$ for $t > 0$.

8. Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

9. Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that

(i) f is a polynomial and

(ii) the degree of f is at least N.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called *univalent*) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

11. Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

    1. Show that g has a unique fixed point in $|z| < 1$.

    2. What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.

    3. What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

    Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

12. Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

13. Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

    (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \ .$$

    (2) Deduce Liouville's theorem from (1).

14. Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

15. Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

16. Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

17. Assume f is continuous in the region: $0 < |z - a| \leq R, \ 0 \leq \arg(z - a) \leq \beta_0 \ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z \to a} (z - a) f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

    where $\gamma_r := \{z \mid z = a + re^{it}, \ 0 \leq t \leq \beta_0\}$.

18. Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

19. (1) Show that the function $u = u(x, y)$ given by

$$u(x,y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

is the solution on $D = \{(x, y) \; |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

(2) Show that there exist points $(x, y) \in D$ such that $\limsup_{n \longrightarrow \infty} |u(x, y)| = \infty$.

2 Fall 2011

1. (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \; .$$

(2) Deduce Liouville's theorem from (1).

2. Let f be a continuous function in the region

$$D = \{z \; ||z| > R, 0 \le \arg Z \le \theta\} \quad \text{where} \quad 0 \le \theta \le 2\pi.$$

If there exists k such that $\lim_{z \longrightarrow \infty} zf(z) = k$ for z in the region D. Show that

$$\lim_{R' \longrightarrow \infty} \int_L f(z) dz = i\theta k,$$

where L is the part of the circle $|z| = R'$ which lies in the region D.

3. Suppose that f is an analytic function in the region D which contains the point a. Let

$$F(z) = z - a - qf(z), \quad \text{where} \quad q \text{ is a complex parameter.}$$

(1) Let $K \subset D$ be a circle with the center at point a and also we assume that $f(z) \neq 0$ for $z \in K$. Prove that the function F has one and only one zero $z = w$ on the closed disc $\overline{K}$ whose boundary is the circle K if $|q| < \min_{z \in K} \dfrac{|z - a|}{|f(z)|}$.

(2) Let $G(z)$ be an analytic function on the disk $\overline{K}$. Apply the residue theorem to prove that $\dfrac{G(w)}{F'(w)} = \dfrac{1}{2\pi i} \displaystyle\int_K \frac{G(z)}{F(z)} dz$, where w is the zero from (1).

(3) If $z \in K$, prove that the function $\dfrac{1}{F(z)}$ can be represented as a convergent series with respect to q: $\dfrac{1}{F(z)} = \displaystyle\sum_{n=0}^{\infty} \frac{(qf(z))^n}{(z-a)^{n+1}}$.

4. Evaluate $\displaystyle\int_0^{\infty} \frac{x \sin x}{x^2+a^2}\, dx$.

5. Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

6. Show that $\displaystyle\int_0^{\infty} \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

7. For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \displaystyle\int_0^{\infty} e^{-t} t^{s-1} dt$.

   1. Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.
   2. Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t \displaystyle\int_0^{\infty} e^{-vt}(vt)^{-s} dv$ for $t > 0$.

8. Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbb{C}$.

9. Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

11. Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.

   - Show that g has a unique fixed point in $|z| < 1$.
   - What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
   - What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha} z}$ may be useful.

12. Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

13. Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

14. Let $f(z)$ be entire and assume values of $f(z)$ lie outside a *bounded* open set Ω. Show without using Picard's theorems that $f(z)$ is a constant.

15. (1) Assume $f(z) = \sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \ .$$

    (2) Deduce Liouville's theorem from (1).

16. Let $f(z)$ be entire and assume that $f(z) \leq M|z|^2$ outside some disk for some constant M. Show that $f(z)$ is a polynomial in z of degree ≤ 2.

17. Let $a_n(z)$ be an analytic sequence in a domain D such that $\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

18. Let $f(z)$ be analytic in an open set Ω except possibly at a point z_0 inside Ω. Show that if $f(z)$ is bounded in near z_0, then $\int_\Delta f(z)dz = 0$ for all triangles Δ in Ω.

19. Assume f is continuous in the region: $0 < |z - a| \leq R, \ 0 \leq \arg(z - a) \leq \beta_0 \ (0 < \beta_0 \leq 2\pi)$ and the limit $\lim_{z \to a} (z - a) f(z) = A$ exists. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f(z)dz = iA\beta_0 \ ,$$

    where $\gamma_r := \{z \mid z = a + re^{it}, \ 0 \leq t \leq \beta_0\}$.

20. Show that $f(z) = z^2$ is uniformly continuous in any open disk $|z| < R$, where $R > 0$ is fixed, but it is not uniformly continuous on $\mathbb{C}$.

    (1) Show that the function $u = u(x, y)$ given by

$$u(x, y) = \frac{e^{ny} - e^{-ny}}{2n^2} \sin nx \quad \text{for } n \in \mathbf{N}$$

    is the solution on $D = \{(x, y) \ |x^2 + y^2 < 1\}$ of the Cauchy problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = \frac{\sin nx}{n}.$$

    (2) Show that there exist points $(x, y) \in D$ such that $\limsup_{n \longrightarrow \infty} |u(x, y)| = \infty$.

3 Spring 2014

1. The question provides some insight into Cauchy's theorem. Solve the problem without using the Cauchy theorem.

   1. Evaluate the integral $\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.
   2. Same question as (a), but with γ any circle not containing the origin.
   3. Show that if $|a| < r < |b|$, then $\int_\gamma \frac{dz}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$. Here γ denotes the circle centered at the origin, of radius r, with the positive orientation.

2. (1) Assume the infinite series $\sum_{n=0}^{\infty} c_n z^n$ converges in $|z| < R$ and let $f(z)$ be the limit. Show that for $r < R$,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} .$$

   (2) Deduce Liouville's theorem from (1). Liouville's theorem: If $f(z)$ is entire and bounded, then f is constant.

3. Let f be a continuous function in the region

$$D = \{z \,||z| > R, 0 \le \arg Z \le \theta\} \quad \text{where} \quad 0 \le \theta \le 2\pi.$$

   If there exists k such that $\lim_{z \longrightarrow \infty} z f(z) = k$ for z in the region D. Show that

$$\lim_{R' \longrightarrow \infty} \int_L f(z) dz = i\theta k,$$

   where L is the part of the circle $|z| = R'$ which lies in the region D.

4. Evaluate $\int_0^\infty \frac{x \sin x}{x^2 + a^2} \, dx$.

5. Let $f = u + iv$ be differentiable (i.e. $f'(z)$ exists) with continuous partial derivatives at a point $z = re^{i\theta}$, $r \neq 0$. Show that

$$\frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}.$$

6. Show that $\int_0^\infty \frac{x^{a-1}}{1+x^n} dx = \frac{\pi}{n \sin \frac{a\pi}{n}}$ using complex analysis, $0 < a < n$. Here n is a positive integer.

7. For $s > 0$, the **gamma function** is defined by $\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt$.

   - Show that the gamma function is analytic in the half-plane $\Re(s) > 0$, and is still given there by the integral formula above.

- Apply the formula in the previous question to show that

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Hint: You may need $\Gamma(1-s) = t\int_0^\infty e^{-vt}(vt)^{-s}dv$ for $t > 0$.

8. Apply Rouché's Theorem to prove the Fundamental Theorem of Algebra: If

$$P_n(z) = a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + a_n z^n \quad (a_n \neq 0)$$

is a polynomial of degree n, then it has n zeros in $\mathbf{C}$.

9. Suppose f is entire and there exist $A, R > 0$ and natural number N such that

$$|f(z)| \geq A|z|^N \text{ for } |z| \geq R.$$

Show that (i) f is a polynomial and (ii) the degree of f is at least N.

10. Let $f : \mathbb{C} \to \mathbb{C}$ be an injective analytic (also called univalent) function. Show that there exist complex numbers $a \neq 0$ and b such that $f(z) = az + b$.

11. Let g be analytic for $|z| \leq 1$ and $|g(z)| < 1$ for $|z| = 1$.
   - Show that g has a unique fixed point in $|z| < 1$.
   - What happens if we replace $|g(z)| < 1$ with $|g(z)| \leq 1$ for $|z| = 1$? Give an example if (a) is not true or give an proof if (a) is still true.
   - What happens if we simply assume that f is analytic for $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$? Suppose that $f(z) \not\equiv z$. Can f have more than one fixed point in $|z| < 1$?

Hint: The map $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ may be useful.

12. Find a conformal map from $D = \{z : \ |z| < 1, \ |z - 1/2| > 1/2\}$ to the unit disk $\Delta = \{z : |z| < 1\}$.

4 Fall 2015

1. Let $a_n \neq 0$ and assume that $\lim_{n\to\infty} \dfrac{|a_{n+1}|}{|a_n|} = L$. Show that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$. In particular, this shows that when applicable, the ratio test can be used to calculate the radius of convergence of a power series.

2. (a) Let z, w be complex numbers, such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$

and also that

$$\left|\frac{w - z}{1 - \overline{w}z}\right| = 1 \quad \text{if } |z| = 1 \text{ or } |w| = 1.$$

(b) Prove that for fixed w in the unit disk $\mathbb{D}$, the mapping

$$F : z \mapsto \frac{w-z}{1-\overline{w}z}$$

satisfies the following conditions:

(c) F maps $\mathbb{D}$ to itself and is holomorphic.

(ii) F interchanges 0 and w, namely, $F(0) = w$ and $F(w) = 0$.

(iii) $|F(z)| = 1$ if $|z| = 1$.

(iv) $F : \mathbb{D} \mapsto \mathbb{D}$ is bijective.

Hint: Calculate $F \circ F$.

3. Use n-th roots of unity (i.e. solutions of $z^n - 1 = 0$) to show that

$$2^{n-1} \sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = n\ .$$

Hint: $1 - \cos 2\theta = 2\sin^2\theta,\ \sin 2\theta = 2\sin\theta\cos\theta.$

(a) Show that in polar coordinates, the Cauchy-Riemann equations take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

(b) Use these equations to show that the logarithm function defined by

$$\log z = \log r + i\theta \ \text{ where } z = re^{i\theta} \text{ with } -\pi < \theta < \pi$$

is a holomorphic function in the region $r > 0,\ -\pi < \theta < \pi$. Also show that $\log z$ defined above is not continuous in $r > 0$.

4. Assume f is continuous in the region: $x \geq x_0,\ 0 \leq y \leq b$ and the limit

$$\lim_{x\to+\infty} f(x+iy) = A$$

exists uniformly with respect to y (independent of y). Show that

$$\lim_{x\to+\infty} \int_{\gamma_x} f(z)dz = iAb\ ,$$

where $\gamma_x := \{z \mid z = x + it,\ 0 \leq t \leq b\}$.

5. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2. Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim\limits_{z\to\infty} f(z) = A$. Show that

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi - z}\,d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$

6. Let $f(z)$ be bounded and analytic in $\mathbb{C}$. Let $a \neq b$ be any fixed complex numbers. Show that the following limit exists

$$\lim_{R\to\infty} \int_{|z|=R} \frac{f(z)}{(z-a)(z-b)} dz.$$

Use this to show that $f(z)$ must be a constant (Liouville's theorem).

7. Prove by *justifying all steps* that for all $\xi \in \mathbb{C}$ we have $e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$.

   Hint: You may use that fact in Example 1 on p. 42 of the textbook without proof, i.e., you may assume the above is true for real values of ξ.

8. Suppose that f is holomorphic in an open set containing the closed unit disc, except for a pole at z_0 on the unit circle. Let denote the the power series in the open disc. Show that (1) $c_n \neq 0$ for all large enough n's, and (2) $\lim_{n\to\infty} \frac{c_n}{c_{n+1}} = z_0$.

9. Let $f(z)$ be a non-constant analytic function in $|z| > 0$ such that $f(z_n) = 0$ for infinite many points z_n with $\lim_{n\to\infty} z_n = 0$. Show that $z = 0$ is an essential singularity for $f(z)$. (An example of such a function is $f(z) = \sin(1/z)$.)

10. Let f be entire and suppose that $\lim_{z\to\infty} f(z) = \infty$. Show that f is a polynomial.

11. Expand the following functions into Laurent series in the indicated regions:

    (a) $f(z) = \frac{z^2-1}{(z+2)(z+3)}$, $2 < |z| < 3$, $3 < |z| < +\infty$.

    (b) $f(z) = \sin \frac{z}{1-z}$, $0 < |z-1| < +\infty$

12. Assume $f(z)$ is analytic in region D and Γ is a rectifiable curve in D with interior in D. Prove that if $f(z)$ is real for all $z \in \Gamma$, then $f(z)$ is a constant.

13. Find the number of roots of $z^4 - 6z + 3 = 0$ in $|z| < 1$ and $1 < |z| < 2$ respectively.

14. Prove that $z^4 + 2z^3 - 2z + 10 = 0$ has exactly one root in each open quadrant.

15. (1) Let $f(z) \in H(\mathbb{D})$, $\mathrm{Re}(f(z)) > 0$, $f(0) = a > 0$. Show that

$$|\frac{f(z)-a}{f(z)+a}| \leq |z|, \quad |f'(0)| \leq 2a.$$

    (2) Show that the above is still true if $\mathrm{Re}(f(z)) > 0$ is replaced with $\mathrm{Re}(f(z)) \geq 0$.

16. Assume $f(z)$ is analytic in $\mathbb{D}$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

17. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r < 1$, let D_r be the disk

$|z| < r$. Show that the area of $f(D_r)$ is finite and is given by

$$S = \pi \sum_{n=1}^{\infty} n|c_n|^2 r^{2n}.$$

(Note that in general the area of $f(D_1)$ is infinite.)

18. Let $f(z) = \displaystyle\sum_{n=-\infty}^{\infty} c_n z^n$ be analytic and one-to-one in $r_0 < |z| < R_0$. For $r_0 < r < R < R_0$, let $D(r, R)$ be the annulus $r < |z| < R$. Show that the area of $f(D(r, R))$ is finite and is given by

$$S = \pi \sum_{n=-\infty}^{\infty} n|c_n|^2 (R^{2n} - r^{2n}).$$

5 Spring 2015

1. Let $a_n(z)$ be an analytic sequence in a domain D such that $\displaystyle\sum_{n=0}^{\infty} |a_n(z)|$ converges uniformly on bounded and closed sub-regions of D. Show that $\displaystyle\sum_{n=0}^{\infty} |a_n'(z)|$ converges uniformly on bounded and closed sub-regions of D.

2. Let f_n, f be analytic functions on the unit disk $\mathbb{D}$. Show that the following are equivalent.
   (i) $f_n(z)$ converges to $f(z)$ uniformly on compact subsets in $\mathbb{D}$.
   (ii) $\displaystyle\int_{|z|=r} |f_n(z) - f(z)|\,|dz|$ converges to 0 if $0 < r < 1$.

3. Let f and g be non-zero analytic functions on a region Ω. Assume $|f(z)| = |g(z)|$ for all z in Ω. Show that $f(z) = e^{i\theta} g(z)$ in Ω for some $0 \le \theta < 2\pi$.

4. Suppose f is analytic in an open set containing the unit disc $\mathbb{D}$ and $|f(z)| = 1$ when $|z|$=1. Show that either $f(z) = e^{i\theta}$ for some $\theta \in \mathbb{R}$ or there are finite number of $z_k \in \mathbb{D}$, $k \le n$ and $\theta \in \mathbb{R}$ such that $f(z) = e^{i\theta} \displaystyle\prod_{k=1}^{n} \frac{z - z_k}{1 - \bar{z}_k z}$.

   Also cf. Stein et al, 1.4.7, 3.8.17

5. (1) Let $p(z)$ be a polynomial, $R > 0$ any positive number, and $m \ge 1$ an integer. Let $M_R = \sup\{|z^m p(z) - 1| : |z| = R\}$. Show that $M_R > 1$.
   (2) Let $m \ge 1$ be an integer and $K = \{z \in \mathbb{C} : r \le |z| \le R\}$ where $r < R$. Show (i) using (1) as well as, (ii) without using (1) that there exists a positive number $\varepsilon_0 > 0$ such that for each polynomial $p(z)$,

   $$\sup\{|p(z) - z^{-m}| : z \in K\} \ge \varepsilon_0\,.$$

6. Let $f(z) = \dfrac{1}{z} + \dfrac{1}{z^2 - 1}$. Find all the Laurent series of f and describe the largest annuli in which these series are valid.

7. Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \leq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at most N.

8. Suppose f is entire and there exist $A, R > 0$ and natural number N such that $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N.

9. (1) Explicitly write down an example of a non-zero analytic function in $|z| < 1$ which has infinitely zeros in $|z| < 1$.

   (2) Why does not the phenomenon in (1) contradict the uniqueness theorem?

10. (1) Assume u is harmonic on open set O and z_n is a sequence in O such that $u(z_n) = 0$ and $\lim z_n \in O$. Prove or disprove that u is identically zero. What if O is a region?

    (2) Assume u is harmonic on open set O and $u(z) = 0$ on a disc in O. Prove or disprove that u is identically zero. What if O is a region?

    (3) Formulate and prove a Schwarz reflection principle for harmonic functions

    cf. Theorem 5.6 on p.60 of Stein et al.

    Hint: Verify the mean value property for your new function obtained by Schwarz reflection principle.

11. Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that

$$||f||_{(\infty,s)} \leq c||f||_{(1,r)},$$

where $|f||_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $||f||_{(1,r)} = \int_{D_r(z_0)} |f(z)|dxdy$.

    Note: Exercise 3.8.20 on p.107 in Stein et al is a straightforward consequence of this stronger result using the integral form of the Cauchy-Schwarz inequality in real analysis.

12. (1) Let f be analytic in $\Omega : 0 < |z - a| < r$ except at a sequence of poles $a_n \in \Omega$ with $\lim\limits_{n\to\infty} a_n = a$. Show that for any $w \in \mathbb{C}$, there exists a sequence $z_n \in \Omega$ such that $\lim\limits_{n\to\infty} f(z_n) = w$.

    (2) Explain the similarity and difference between the above assertion and the Weierstrass-Casorati theorem.

13. Compute the following integrals.

    (i) $\displaystyle\int_0^\infty \frac{1}{(1+x^n)^2}\,dx$, $n \geq 1$ (ii) $\displaystyle\int_0^\infty \frac{\cos x}{(x^2+a^2)^2}\,dx$, $a \in \mathbb{R}$ (iii) $\displaystyle\int_0^\pi \frac{1}{a+\sin\theta}\,d\theta$, $a > 1$

    (iv) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{a+\sin^2\theta}$, $a > 0$. (v) $\displaystyle\int_{|z|=2} \frac{1}{(z^5-1)(z-3)}\,dz$ (v) $\displaystyle\int_{-\infty}^\infty \frac{\sin\pi a}{\cosh\pi x+\cos\pi a} e^{-ix\xi}\,dx$, $0 < a < 1$, $\xi \in \mathbb{R}$ (vi) $\displaystyle\int_{|z|=1} \cot^2 z\,dz$.

14. Compute the following integrals.

    (i) $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ (ii) $\displaystyle\int_0^\infty (\frac{\sin x}{x})^2\,dx$ (iii) $\displaystyle\int_0^\infty \frac{x^{a-1}}{(1+x)^2}\,dx$, $0 < a < 2$

(i) $\displaystyle\int_0^\infty \frac{\cos ax - \cos bx}{x^2} dx,\ a, b > 0$ (ii) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\, dx,\ 0 < a < n$

(iii) $\displaystyle\int_0^\infty \frac{\log x}{1+x^n}\, dx,\ n \geq 2$ (iv) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$ (v) $\displaystyle\int_0^\pi \log|1 - a\sin\theta| d\theta,\ a \in \mathbb{C}$

15. Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

16. Let f be an analytic function on a region Ω. Show that f is a constant if there is a simple closed curve γ in Ω such that its image $f(\gamma)$ is contained in the real axis.

17. (1) Show that $\dfrac{\pi^2}{\sin^2 \pi z}$ and $g(z) = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ have the same principal part at each integer point.

    (2) Show that $h(z) = \dfrac{\pi^2}{\sin^2 \pi z} - g(z)$ is bounded on $\mathbb{C}$ and conclude that $\dfrac{\pi^2}{\sin^2 \pi z} = \displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$.

18. Let $f(z)$ be an analytic function on $\mathbb{C}\backslash\{z_0\}$, where z_0 is a fixed point. Assume that $f(z)$ is bijective from $\mathbb{C}\backslash\{z_0\}$ onto its image, and that $f(z)$ is bounded outside $D_r(z_0)$, where r is some fixed positive number. Show that there exist $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$, $c \neq 0$ such that $f(z) = \dfrac{az+b}{cz+d}$.

19. Assume $f(z)$ is analytic in $\mathbb{D} : |z| < 1$ and $f(0) = 0$ and is not a rotation (i.e. $f(z) \neq e^{i\theta}z$). Show that $\displaystyle\sum_{n=1}^{\infty} f^n(z)$ converges uniformly to an analytic function on compact subsets of $\mathbb{D}$, where $f^{n+1}(z) = f(f^n(z))$.

20. Let f be a non-constant analytic function on $\mathbb{D}$ with $f(\mathbb{D}) \subseteq \mathbb{D}$. Use $\psi_a(f(z))$ (where $a = f(0)$, $\psi_a(z) = \dfrac{a-z}{1-\bar{a}z}$) to prove that $\dfrac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \dfrac{|f(0)| + |z|}{1 - |f(0)||z|}$.

21. Find a conformal map
    1. from $\{z : |z - 1/2| > 1/2, \mathrm{Re}(z) > 0\}$ to $\mathbb{H}$
    2. from $\{z : |z - 1/2| > 1/2, |z| < 1\}$ to $\mathbb{D}$
    3. from the intersection of the disk $|z + i| < \sqrt{2}$ with $\mathbb{H}$ to $\mathbb{D}$.
    4. from $\mathbb{D}\backslash[a, 1)$ to $\mathbb{D}\backslash[0, 1)$ $(0 < a < 1)$. Short solution possible using Blaschke factor
    5. from $\{z : |z| < 1, \mathrm{Re}(z) > 0\}\backslash(0, 1/2]$ to $\mathbb{H}$.

22. Let C and C' be two circles and let $z_1 \in C$, $z_2 \notin C$, $z_1' \in C'$, $z_2' \notin C'$. Show that there is a unique fractional linear transformation f with $f(C) = C'$ and $f(z_1) = z_1'$, $f(z_2) = z_2'$.

23. Assume $f_n \in H(\Omega)$ is a sequence of holomorphic functions on the region Ω that are uniformly bounded on compact subsets and $f \in H(\Omega)$ is such that the set $\{z \in \Omega : \lim_{n\to\infty} f_n(z) = f(z)\}$ has a limit point in Ω. Show that f_n converges to f uniformly on compact subsets of Ω.

24. Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1 - \bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : \ |z| < 1\}$. Prove that

- $\displaystyle \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1.$
- $\displaystyle \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$

25. Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from half disc $\{z = x+iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

26. Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \le M$ for all $z \in \gamma$. Prove that $|f(z)| \le M$ for all $z \in U$.

27. Compute the following integrals. (i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx$, $0 < a < n$ (ii) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

28. Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

29. Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that
$$\|f\|_{(\infty,s)} \le c\|f\|_{(1,r)},$$
where $\|f\|_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $\|f\|_{(1,r)} = \displaystyle\int_{D_r(z_0)} |f(z)|dxdy.$

30. Let $\psi_\alpha(z) = \dfrac{\alpha - z}{1-\bar{\alpha}z}$ with $|\alpha| < 1$ and $\mathbb{D} = \{z : |z| < 1\}$. Prove that
    - $\displaystyle \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha|^2 dxdy = 1.$
    - $\displaystyle \frac{1}{\pi}\iint_{\mathbb{D}} |\psi'_\alpha| dxdy = \frac{1-|\alpha|^2}{|\alpha|^2}\log\frac{1}{1-|\alpha|^2}.$

    Prove that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from half disc $\{z = x+iy : |z| < 1, y > 0\}$ to upper half plane $\mathbb{H} = \{z = x + iy : y > 0\}$.

31. Let Ω be a simply connected open set and let γ be a simple closed contour in Ω and enclosing a bounded region U anticlockwise. Let $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function and $|f(z)| \le M$ for all $z \in \gamma$. Prove that $|f(z)| \le M$ for all $z \in U$.

32. Compute the following integrals. (i) $\displaystyle\int_0^\infty \frac{x^{a-1}}{1+x^n}\,dx$, $0 < a < n$

    (ii) $\displaystyle\int_0^\infty \frac{\log x}{(1+x^2)^2}\,dx$

33. Let $0 < r < 1$. Show that polynomials $P_n(z) = 1 + 2z + 3z^2 + \cdots + nz^{n-1}$ have no zeros in $|z| < r$ for all sufficiently large n's.

34. Let f be holomorphic in a neighborhood of $D_r(z_0)$. Show that for any $s < r$, there exists a constant $c > 0$ such that
$$\|f\|_{(\infty,s)} \le c\|f\|_{(1,r)},$$

where $\|f\|_{(\infty,s)} = \sup_{z\in D_s(z_0)}|f(z)|$ and $\|f\|_{(1,r)} = \int_{D_r(z_0)} |f(z)|dxdy$.

6 Fall 2016

1. Let $u(x, y)$ be harmonic and have continuous partial derivatives of order three in an open disc of radius $R > 0$.

   (a) Let two points $(a, b), (x, y)$ in this disk be given. Show that the following integral is independent of the path in this disk joining these points:

   $$v(x, y) = \int_{a,b}^{x,y} (-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy).$$

   (b)

   (i) Prove that $u(x, y) + iv(x, y)$ is an analytic function in this disc.

   (ii) Prove that $v(x, y)$ is harmonic in this disc.

2. (a) $f(z) = u(x, y) + iv(x, y)$ be analytic in a domain $D \subset \mathbb{C}$. Let $z_0 = (x_0, y_0)$ be a point in D which is in the intersection of the curves $u(x, y) = c_1$ and $v(x, y) = c_2$, where c_1 and c_2 are constants. Suppose that $f'(z_0) \neq 0$. Prove that the lines tangent to these curves at z_0 are perpendicular.

   (b) Let $f(z) = z^2$ be defined in $\mathbb{C}$.

   (c) Describe the level curves of $\text{Re}(f)$ and of $\text{Im}(f)$.

   (ii) What are the angles of intersections between the level curves $\text{Re}(f) = 0$ and $\text{Im}(f)$? Is your answer in agreement with part a) of this question?

3. (a) $f : D \to \mathbb{C}$ be a continuous function, where $D \subset \mathbb{C}$ is a domain.Let $\alpha : [a, b] \to D$ be a smooth curve. Give a precise definition of the *complex line integral*

   $$\int_\alpha f.$$

   (b) Assume that there exists a constant M such that $|f(\tau)| \leq M$ for all $\tau \in \text{Image}(\alpha)$. Prove that

   $$|\int_\alpha f| \leq M \times \text{length}(\alpha).$$

   (c) Let C_R be the circle $|z| = R$, described in the counterclockwise direction, where $R > 1$. Provide an upper bound for $|\int_{C_R} \frac{\log(z)}{z^2}|$, which depends only on R and other constants.

4. (a) Let Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume the existence of a non-negative integer m, and of positive constants L and R, such that for all z with $|z| > R$ the inequality

   $$|f(z)| \leq L|z|^m$$

   holds. Prove that f is a polynomial of degree $\leq m$.

(b) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Suppose that there exists a real number M such that for all $z \in \mathbb{C}$

$$\mathrm{Re}(f) \leq M.$$

Prove that f must be a constant.

5. Prove that all the roots of the complex polynomial

$$z^7 - 5z^3 + 12 = 0$$

lie between the circles $|z| = 1$ and $|z| = 2$.

6. (a) Let F be an analytic function inside and on a simple closed curve C, except for a pole of order $m \geq 1$ at $z = a$ inside C. Prove that

$$\frac{1}{2\pi i} \oint_C F(\tau) d\tau = \lim_{\tau \to a} \frac{d^{m-1}}{d\tau^{m-1}} ((\tau - a)^m F(\tau)).$$

(b) Evaluate

$$\oint_C \frac{e^\tau}{(\tau^2 + \pi^2)^2} d\tau$$

where C is the circle $|z| = 4$.

7. Find the conformal map that takes the upper half-plane comformally onto the half-strip $\{w = x + iy : \ -\pi/2 < x < \pi/2 \ y > 0\}$.

8. Compute the integral $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} dx$ where $\cosh z = \dfrac{e^z + e^{-z}}{2}$.