# Solutions to Texas A&M's Real Analysis Qual Courses

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These are the solutions to the majority of the available past real qualifying exams for Texas A&M. Incomplete/non-existent solutions are marked in red. If you find any errors/typos or have solutions to the unsolved questions, please email me at keifer@math.tamu.edu.

## Contents

# 1 August 2019

**Problem 1.** *Let $(X, \mathcal{M}, \mu)$ be a measure space and $f$ a measurable non-negative function on $X$. Define $\nu : \mathcal{M} \to [0, \infty]$ by*

$$\nu(E) = \int_E f d\mu.$$

*(a) Prove that $\nu$ is a measure*

*Proof.* Indeed, it's clear that $\nu(E) = \int_E f d\mu \geq 0$ for all $E$ since $f$ is assumed to be non-negative. It's equally clear that $\nu(\emptyset) = \int_\emptyset f d\mu = 0$.

We are only left to prove countable additivity. Take a countable collection $\{E_i\}$ of pairwise disjoint sets in $\mathcal{M}$, so we see for finitely many

$$\nu\left(\cup_{k=1}^N E_k\right) = \int_{\cup_{k=1}^N E_k} f d\mu = \int \chi_{\cup_{k=1}^N E_k} f d\mu = \sum_{k=1}^N \int \chi_{E_k} f d\mu = \sum_{k=1}^N \int_{E_k} f d\mu = \sum_{k=1}^N \nu(E_k).$$

By the monotone convergence theorem (the finite sums of characteristic functions form an increasing sequence that converges to the infinite sum pointwise), then $\nu$ is countably additive. Hence, $\nu$ is a measure. □

*(b) Prove that $g \in L^1(\nu)$ if and only if $gf \in L^1(\mu)$ and in that case $\int_X g d\nu = \int_X gf d\mu$.*

*Proof.* First we show that $\nu \ll \mu$. Indeed, if $\mu(E) = 0$ then choose an increasing sequence of simple functions $f_n$ such that $f_n \to f$. Then by monotone convergence theorem and the definition of integral for simple functions, we have

$$\nu(E) = \int_E f d\mu = \int_E (\lim f_n) d\mu = \lim \int_E f_n d\mu = 0.$$

Then we may apply Radon-Nikodym theorem to see that $f = \frac{d\nu}{d\mu}$ and see that $g \in L^1(\nu)$ if and only if $\int_X |g| d\nu < \infty$ which is equivalent to $\int_X |g| f d\mu = \int_X |g| \frac{d\nu}{d\mu} d\mu < \infty$. Since $f$ is non-negative, this is equivalent to having $\int_X |gf| d\mu < \infty$. Radon-Nikodym also tells us that $\int_X g d\nu = \int_X gf d\mu$. □

**Problem 2.** *(a) State Fatou's lemma*

*Proof.* For $f_n \in L^+$ then

$$\int \liminf f_n \leq \liminf \int f_n$$

□

*(b) State the dominated convergence theorem*

*Proof.* Let $g, g_n \in L^+$ be measurable, $|f_n| \leq g_n$ $\mu$-a.e., $f_n \to f$ and $g_n \to f$ $\mu$-a.e. with $\int g_n \to \int g < \infty$. Then $\int f_n \to \int f$. Moreover, $\int |f - f_n| \to 0$. □

(c) *Let $f_n, g_n, h_n, f, g, h$ be measurable functions on $\mathbb{R}^n$ satisfying $f_n \leq g_n \leq h_n$, $f_n \to f$ a.e., $g_n \to g$ a.e., and $h_n \to h$ a.e. Suppose moreover that $f, h \in L^1$ and $\int f_n \to \int f$, $\int h_n \to \int h$. Prove that $g \in L^1$ and $\int g_n \to \int g$.*

*Proof.* here □

**Problem 3.** *Let $\{A_k\}_{k=1}^{\infty}$ be measurable subsets of a measure space and define $B_m$ to be the set of all points which are contained in at least $m$ of the sets $\{A_k\}_{k=1}^{\infty}$. Prove that $B_m$ is measurable and*

$$\mu(B_m) \leq \frac{1}{m} \sum_{k=1}^{\infty} \mu(A_k).$$

*Proof.* Let $C = \{F \subseteq \mathbb{N} \mid |F| = m\}$ which is a countable infinite set. Then we may express

$$B_m = \bigcup_{F \in C} \bigcap_{i \in F} A_i.$$

Therefore, each $B_m$ is measurable.

here □

**Problem 4.** *Let $E$ be a subset of $\mathbb{R}$ which is not Lebesgue measurable. Prove that there exists an $\eta > 0$ such that for any two Lebesgue measurable sets $A, B$ satisfying $A \subseteq E \subseteq B$ one has $\lambda(B \backslash A) > \eta$, where $\lambda$ denotes Lebesgue measure.*

*Proof.* here □

**Problem 5.** *Let $\{A_k\}_{k=1}^{\infty}$ be Lebesgue measurable sets in $\mathbb{R}^n$ equipped with Lebesgue measure $\lambda$.*

(a) *Prove that if $A_k \subseteq A_{k+1}$ for all $k$ then $\lambda(\cup_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \lambda(A_k)$*

*Proof.* We will assume that $\lambda$ is subadditive, so $\lambda(\cup_1^\infty A_k) \leq \sum_1^\infty \lambda(A_k)$.Then by setting $A_k = \emptyset$, we have

$$\lambda\left(\cup_1^\infty A_k\right) = \sum_1^\infty \lambda(A_j \backslash A_{j-1}) = \lim_{n\to\infty} \sum_1^n \lambda(A_j \backslash A_{j-1}) = \lim_{n\to\infty} \lambda(A_n).$$

□

(b) *Prove that if $A_{k+1} \subseteq A_k$ for all $k$ and $\lambda(A_1) < \infty$ then $\lambda(\cap_{k=1}^{\infty} A_k) = \lim_{k\to\infty} \lambda(A_k)$*

*Proof.* Let $B_j = A_1 \backslash A_j$ so $B_1 \subset B_2 \subset \ldots$, and $\lambda(A_1) = \lambda(B_j) + \lambda(A_j)$, and $\cup_1^\infty B_j = E_1 \backslash (\cap_1^\infty A_j)$. Then by part (a), we have

$$\lambda(A_1) = \lambda\left(\bigcap_1^\infty A_j\right) + \lim_{j\to\infty} \lambda(B_j) = \lambda\left(\cap_1^\infty A_j\right) + \lim_{j\to\infty} (\lambda(A_1) - \lambda(A_j)).$$

Since $\lambda(A_1) < \infty$, we may subtract it from both sides to yield the desired result. □

*(c) Give an example showing that without assuming $\lambda(A_1) < \infty$ the conclusion of the previous part does not hold.*

*Proof.* Consider $A_j = [j, \infty)$ so that for each $j$, $\lambda(A_j) = \infty$ but $\cap_1^\infty A_j = \emptyset$ so $\lambda(\cap_1^\infty A_j) = 0$. □

**Problem 6.** *Let $X$ and $Y$ be Banach spaces. Show that the linear space $X \oplus Y$ is a Banach space under the norm $\|(x, y)\| = \|x\| + \|y\|$. Also determine (with justification) the dual $(X \oplus Y)^*$.*

*Proof.* here □

**Problem 7.** *For each $n \in \mathbb{N}$ define on $\ell^\infty$ the linear functional $\varphi_n(x) = n^{-1}\sum_{k=1}^n x(k)$. Let $\varphi$ be the weak* cluster point of the sequence $\{\varphi_n\}$. Show that $\varphi$ does not belong to the image of $\ell^1$ under the canonical embedding $\ell^1 \hookrightarrow (\ell^\infty)^*$.*

*Proof.* here □

**Problem 8.** *Let $T : X \to Y$ be a surjective linear map between Banach spaces and suppose that there is a $\lambda > 0$ such that $\|Tx\| \geq \lambda\|x\|$ for all $x \in X$. Show that $T$ is bounded.*

*Proof.* here □

**Problem 9.** *Let $X$ be a compact metric space and $\mu$ a regular Borel measure on $X$. Let $f : X \to [0, \infty)$ be a continuous function and for each $n \in \mathbb{N}$ set $f_n(x) = f(x)^{1/n}$ for all $x \in X$. Show that $\int f_n d\mu \to \mu(\operatorname{supp} f)$ as $n \to \infty$ where $\operatorname{supp} f = \{x \in X \mid f(x) > 0\}$.*

*Proof.* here □

**Problem 10.** *Let $X$ be a compact metric space and let $x \in X$. Suppose that the point mass $\delta_x$ is the weak* limit of a sequence of atomless Radon measures on $X$ (viewing all of these measures as elements of $C(X)^*$). Show that every neighborhood of $x$ is uncountable.*

*Proof.* here □

# 2 January 2019

**Problem 1.** *True or false (prove or give a counter example)*

*(a) Let $E \subseteq \mathbb{R}$ be a Borel set, then $\{(x, y) \in \mathbb{R}^2 \mid x - y \in E\}$ is a Borel set in $\mathbb{R}^2$.*

*Proof.* TRUE.

Define $f(x, y) = x - y : \mathbb{R}^2 \to \mathbb{R}$. This is continuous. Let

$$\mathcal{A} := \{S \subseteq \mathbb{R} \mid f^{-1}(S) \text{ is a Borel set of } \mathbb{R}^2\}$$

Then $\mathcal{A}$ is a $\sigma$-algebra (easy to check). If $S$ is open, then $f^{-1}(S)$ is open in $\mathbb{R}^2$, thus Borel. So $\{\text{open sets}\} \subseteq \mathcal{A}$ and so the Borel algebra is a subset of $\mathcal{A}$. In particular, $E \in \mathcal{A}$. □

(b) *Let $E \subseteq Q := [0,1] \times [0,1]$. Assume that for every $x, y \in [0,1]$ the sets $E_x = \{y \in [0,1] \mid (x,y) \in E\}$ and $E^y = \{x \in [0,1] \mid (x,y) \in E\}$ are Borel. Then $E$ is Borel.*

*Proof.* FALSE.

Consider a non-Borel set $A \subset [0,1]$. Set $E = \{(x,x) \mid x \in A\}$. Then each $E^y$ and $E_x$ is a singleton which is Borel, but $E$ is not. □

(c) *A function $f : \mathbb{R} \to \mathbb{R}$ is called Lipschitz if there exits a $M > 0$ such that $\forall x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq M|x - y|$. If $A \subseteq \mathbb{R}$ is Lebesgue measureable and $f$ is Lipschitz then $f(A)$ is Lebesgue measurable.*

*Proof.* TRUE.

Since $A$ is Lebesgue measurable, then we can write $A = (\cup_j K_j) \cup N$ where each $K_j$ is a compact set and $N$ has Lebesgue measure zero. Then $f(A) = (\cup_j f(K_j)) \cup f(N)$. It's clear that each $f(K_j)$ is Lebesgue measurable, since $f$ is Lipschitz. We are only left to see that $f(N)$ is also Lebesgue measurable.

Indeed, for every $\epsilon > 0$ we can write $N \subseteq \cup_k B_k$ where each $B_k$ is a ball of radius $r_k$ and $\sum_k m(B_k) < \epsilon$. But then by Lipschitz continuity, $f(B_k)$ is contained in a ball of radius $Mr_k$ where $M$ is the Lipschitz constant of $f$. Thus, $m(f(B_k)) \leq Mm(B_k)$ so that $m(f(N)) \leq M \sum_k m(B_k) < M\epsilon$. Let $\epsilon \to 0$ so $f(N)$ must have outer measure equal to zero, hence it is a null set. □

**Problem 2.** *Let $(X, \mathcal{F}, \mu)$ be a measure space. is it true that for every measurable essentially bounded $f : X \to \mathbb{R}$ we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$? Give an answer both in the case that $\mu$ is finite and the case that $\mu$ is $\sigma$-finite.*

*Proof.* If $\mu$ is finite: By Hölder, we know that $\|f\|_p \leq \|f\|_q$ when $p \leq q$. Also, $\|f\|_p \leq \|f\|_\infty$ for all $p$. Therefore, $\|f\|_p \nearrow \leq \|f\|_\infty$ and so $\lim_p \|f\|_p \leq \|f\|_\infty$.

On the other hand, for every $\epsilon > 0$, let $E = \{x \mid |f(x)| > \|f\|_\infty - \epsilon\}$ and $0 < \mu(E) \leq 1$ since $\|f\|_\infty = \text{esssup}\,|f(x)| < \infty$. Then $\|f\|_p^p \geq \int_E |f|^p > \left(\|f\|_\infty - \epsilon\right)^p \mu(E)$. Take $p \to \infty$ so $\lim_p \|f\|_p \geq \|f\|_\infty - \epsilon$, implying $\lim_p \|f\|_p \geq \|f\|_\infty$.

If $\mu$ is $\sigma$-finite: No, this is not true. Consider $f(x) = \frac{1}{x}$ on $[1, \infty)$. Then $\lim_p \|f\|_p = 0 \neq \|f\|_\infty = 1$. □

**Problem 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ Lebesgue integrable and for $n \in \mathbb{N}$ define*

$$g_n(x) = n \int_{(x, x+1/n)} f \, d\lambda.$$

(a) *Prove that* $\lim_{n\to\infty} g_n = f$ $\lambda$*-a.e.*

*Proof.* This is Lebesgue Differentiation Theorem, with $E_r = (x, x+r)$. □

(b) *Prove that for every* $n \in \mathbb{N}$, $\int_{\mathbb{R}} |g_n| d\lambda \leq \int_{\mathbb{R}} |f| d\lambda$.

*Proof.* here! □

(c) *Prove* $\lim_{n\to\infty} \int_{\mathbb{R}} |g_n| d\lambda = \int_{\mathbb{R}} |f| d\lambda$.

*Proof.* Apply dominated convergence theorem with parts (a) and (b). □

**Problem 4.** *Let* $f \in L^1((0,1]^2, \lambda_2)$ *such that* $\int_{(0,x]\times(0,y]} f d\lambda_2 = 0$ *for every* $x, y \in (0,1]$. *Prove that* $f = 0$ $\lambda_2$*-a.e.*

*Proof.* First note that

$$(a,b) \times (c,d) = \bigcup_n ((0, b-1/n] \times (0, d-1/n]) \setminus ((0,a] \times (0,1] \cup (0,1] \times (0,b])$$

And since all open rectangles generate all Borel sets in $\mathbb{R}^2$, then we have that for every Borel set $B \subseteq \mathbb{R}^2$, $\int_B f d\lambda_2 = 0$.

Since every Lebesgue set $A$ is of the form $A = B \cup N$ where $B$ is a Borel measurable set and $N$ is a set of measure zero. Hence, $\int_A f d\lambda_2 = 0$ for any Lebesgue measurable set $A$.

Now consider $A^+ = \{x \mid f(x) > 0\}$ and $A_- = \{x \mid f(x) < 0\}$. Since both are measurable, then $\int_{A^+} f d\lambda_2 = 0 = \int_{A^-} f d\lambda_2$. Hence, $f = 0$ $\lambda_2$-a.e. □

**Problem 5.** *Let* $\lambda$ *be the Lebesgue measure on* $\mathbb{R}$. *Let* $E \subseteq \mathbb{R}$ *be Lebesgue measurable such that* $0 < \lambda(E) < \infty$. *Prove that for all* $0 \leq \gamma < 1$ *there exists an open interval* $I \subseteq \mathbb{R}$ *such that*

$$\lambda(E \cap I) \geq \gamma\lambda(I).$$

*Proof.* Choose an open set $O \supset E$ such that $\lambda(E) \geq \gamma\lambda(O)$. We can write $O = \cup_i O_I$ for open and disjoint intervals $O_i$. Hence

$$E = E \cap O = E \cap \bigcup_i O_i = \bigcup_i (E \cap O_i)$$

Suppose to the contrary that $\lambda(E \cap O_i) < \gamma\lambda(O_i)$ for all $i$. Then

$$\lambda(E) = \lambda\left(\bigcup(E \cap O_i)\right) = \sum_i \lambda(E \cap O_i) < \gamma \sum_i \lambda(O_i) = \gamma\lambda(O)$$

which is a contradiction with the fact that $\lambda(E) \geq \gamma\lambda(O)$. Hence, it must be that for some $k$, $\lambda(E \cap O_k) \geq \gamma\lambda(O_k)$. □

**Problem 6.** *Let $X$ be a compact metrizable space and $\{\mu_n\}$ a sequence of Borel measures on $X$ with $\mu_n(X) = 1$ for every $n$. Consider the linear map $\varphi : C(X) \to \ell^\infty(\mathbb{N})$ defined by $\varphi(f) = \left(\int_X f d\mu_n\right)_n$. What conditions on the sequence $\{\mu_n\}$ are equivalent to $\varphi$ being an isometry? Provide justification.*

*Proof.* We would require, for all $f \in C(X)$

$$\sup_{x\in X} |f(x)| = \|f\|_\infty = \left\| \left( \int f d\mu_n \right) \right\|_\infty = \sup_n \left| \int f d\mu_n \right|$$

here □

**Problem 7.** *Let $X$ be a compact metric space and $\{f_n\}$ a sequence in $C(X)$. Prove that $\{f_n\}$ converges weakly in $C(X)$ if and only if it converges pointwise and $\sup_n \|f_n\| < \infty$. Also, give an example of an $X$ and a sequence $\{f_n\}$ in $C(X)$ which converges weakly but not uniformly.*

*Proof.* By considering $f_n - f$, we may assume without loss of generality that $f_n$ converges to 0.

$\Rightarrow$) We know $C[0,1]^* = \mathcal{M}[0,1]$. Then $f_n \to 0$ weakly implies $\int f_n d\mu \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Choose $\mu = \delta_t$ so

$$\int f_n d\delta_t = f_n(t) \to 0 \quad \forall t \in [0,1]$$

(this follows from the fact that weak convergence implies uniformly bounded). Consider

$$\begin{aligned} &\chi : C[0,1] \to C[0,1]^{**} = \mathcal{M}[0,1]^* \\ &\chi(f_n)(\mu) = \mu(f_n) \end{aligned}$$

Since $\mu(f_n) \to 0$ then $\chi(f_n)(\mu) \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Since convergent sequences are bounded, then $\sup_n |\chi(f_n)(\mu)| \le M$.

By the uniform boundedness theorem, $\sup_n \|\chi(f_n)\| < \infty$. By isometry, $\|f_n\| = \|\chi(f_n)\|$ so $\sup_n \|f_n\| < \infty$.

$\Leftarrow$) By Dominated Convergence Theorem, $f_n \to 0$ in $L^1(\mu)$. So therefore, $|\int f_n d\mu| \le \int |f_n| d|\mu| \to 0$. So $f_n \to 0$ weakly.

Example: Take $X = [0,1]$ and consider the functions

$$f_n(x) = \begin{cases} nx & x \in [0, 1/n] \\ -nx + 2 & x \in (1/n, 2/n] \\ 0 & x \in (2/n, 1] \end{cases}$$

Then they converge to 0 weakly, but not strongly. □

**Problem 8.** *Let $X$ be a Banach space. Show that if $X^{**}$ is separable then so is $X$. Also, give an example, with justification, to show that the converse is false.*

*Proof.* We will show the weaker result that states that if the dual $X^*$ is separable, then so is $X$.

Let $X^*$ be separable. Consider the unit sphere $S_{X^*} = \{\varphi \in X^* \mid \|\varphi\| = 1\}$. Then $S_{X^*}$ is separable and so we can let $\{\varphi_n\}$ be a countable dense subset of $S_{X^*}$.

For each $n \in \mathbb{N}$, choose $x_n \in \mathbb{N}$ with $\|x_n\| = 1$ such that $|\varphi_n(x_n)| > 1/2$. Let $D = \overline{\text{span}}\{x_1, x_2, \ldots\}$.

Then $D$ is countable; ex. we can consider the following set countable and dense subset of $D$:

$$\bigcup_{n\in\mathbb{N}} \left\{ \sum_{j=1}^{n} (a_j + ib_j)x_j \mid a_j, b_j \in \mathbb{Q} \right\}$$

We want to show that $D = X$. Suppose it were not, then there ewould be some $\varphi \in S_{X^*}$ with $\varphi|_D = 0$. Since $\{\varphi_n\}$ is dense, there exists some $n$ such that $\|\varphi - \varphi_n\| < 1/4$. Therefore,

$$\frac{1}{2} \leq |\varphi_n(x_n)| = |\varphi_n(x_n) - \varphi(x_n)| \leq \|\varphi_n - \varphi\| \|x_n\| < \frac{1}{4}.$$

This is a contradiction and hence, $D = X$.

<u>example.</u> $c_0$ is separable, but $\ell^\infty = c_0^{**}$ is not separable. □

**Problem 9.** *(a) Let $X$ be a compact metrizable space. Describe the dual of $C(X)$ according to the Riesz representation theorem.*

*Proof.* For every $\varphi \in C(X)^*$, there exists a unique finite regular signed measure $\mu$ on the Borel subsets of $X$ such that

$$\varphi(f) = \int_X f d\mu$$

for each $f \in C(X)$. Moreover, $\|\varphi\| = |\mu|(X)$.

□

*(b) Consider the spaces $X = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ and $Y = [0, 1]$ with the topologies inherited from $\mathbb{R}$. Prove that there does not exist a bijective bounded linear map from $C(X)$ to $C(Y)$.*

*Proof.* By contradiction. Suppose there exists a bijective bounded linear map $T : C(X) \to C(Y)$. Then by the Open Mapping Theorem (or more accurately, the corollary that is the Bounded Inverse Theorem), then $T^{-1}$ is a bijective bounded linear map from $C(Y)$ to $C(X)$. This says that the two spaces are isomorphic.

Therefore, the duals of these two spaces should also be isomorphic, $C(X)^* \cong C(Y)^*$. But by the Riesz-representation theorem, here! □

**Problem 10.** *Let $X$ be a Banach space and $Y$ a subspace of $X$. Show that $\|x + Y\| = \inf\{\|x + y\| \mid y \in Y\}$ defines a norm on $X/Y$ if and only if $Y$ is closed.*

*Proof.* $\Leftarrow$) Suppose $Y$ is closed. It's easy to see $\|x+Y\|$ is well-defined and a semi-norm. Suppose $\|x+Y\| = 0$. Then there exists $y_n \in Y$ such that $\|x - y_n\| \to 0$. Since $Y$ is closed, then $x \in Y$. Therefore, $x + Y = Y = 0 + Y$ which is the zero vector in $X/Y$.

$\Rightarrow$) Suppose this is a norm. Take any convergent sequence $y_n$ in $Y$ with $y_n \to y'$. Then $\inf_{y\in Y} \|y - y'\| \leq \|y_n - y'\| \to 0$ and so $\|y' + Y\| = 0$. Since this is a norm, then $y' + Y = 0 + Y = Y$ and so $y' \in Y$. Hence $Y$ must be closed. □

# 3 August 2018

(Solve any 10 of the following 12 problems)

**Problem 1.** *Let $\mu$ and $\nu$ be positive measures on the same measurable space with $\nu$ finite and absolutely continuous with respect to $\mu$. Show that for every $\epsilon > 0$, there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.*

*Proof.* Suppose for contradiction that $\exists \epsilon > 0$ such that $\mu(E) < \delta$ then $\nu(E) \geq \epsilon$ for all $\delta > 0$ adn for some $E$. We'll construct the set $E_n$ to be some set with $\mu(E_n) < 2^{-n}$. Let $F_k = \cup_{n=k}^{\infty} E_n$ so $\mu(F_k) < 2^{-k+1}$.

Let $F = \cap_{k=1}^{\infty} F_k$ so $\mu(F) = 0$. Since $\nu \ll \mu$, then $\nu(F) = 0$.

However, since $F_k$ is a decreasing sequence, we have

$$\nu(F) = \lim_n \nu\left(\cap_{k=1}^{n} F_k\right) = \lim_n \nu(F_n) \geq \epsilon.$$

Contradiction! □

**Problem 2.** *Let $\mu$ be a positive measure. Suppose that $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence in $L^1(\mu)$. Show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $\mu(E) < \delta$ implies*

$$\forall n \geq 1 \qquad \left|\int_E f_n d\mu\right| < \epsilon.$$

*You may use without proof the result of problem #1.*

*Proof.* Let $\epsilon > 0$. Since $\{f_n\}$ is Cauchy in $L^1(\mu)$, there exists $f \in L^1(mu)$ such that $f_n \to f$ in $L^1(\mu)$ as $n \to \infty$, since $L^1(\mu)$ is a Banach space.

Define $\nu(E) := \left|\int_E f d\mu\right|$.

Then by Problem 1, there exists some $\delta > 0$ such that $\nu(E) = \left|\int_E f d\mu\right| < \epsilon/2$ when $\mu(E) < \delta$, thenf or large enough $n$ (say $n \geq N$) we have

$$\left|\int_E f_n d\mu\right| = \left|\int_E (f_n - f + f) d\mu\right| \leq \left|\int_E f_n - f d\mu\right| + \left|\int_E f d\mu\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for $\mu(E) < \delta$.

For each $i, N$, we can find $\delta_i$ such that $\left|\int_E f_i d\mu\right| < \epsilon$ when $\mu(E) < \delta_i$. By the same reasoning as above, if we set $\tilde{\delta} = \min\{\delta_1, \ldots, \delta_{N-1}, \delta\}$ then $\left|\int_E f_n d\mu\right| < \epsilon$ whenever $\mu(E) < \tilde{\delta}$ for all $n \in \mathbb{N}$. □

**Problem 3.** *Let $f : [0,1] \to [0,\infty)$ be Lebesgue measurable. For $n \in \mathbb{N}$ define*

$$g_n = \frac{f^n}{1 + f^n}.$$

*(a) Explain why $\int_0^1 g_n(t)dt$ exists and is finite for all $n$.*

*Proof.* Since $g_n = \frac{f^n}{1+f^n} \leq 1$ for all $n$, then $\int_0^1 g_n dx \leq \int_0^1 1dx = 1$ for all $n$. □

*(b) Prove that $\lim_n \int_0^1 g_n(t)dt$ exists and find an expression for it. Make sure to state which major theorems you are using in your proof.*

*Proof.* Define $E_1 = \{x \mid 0 \leq f(x) < 1\}$, $E_2 = \{x \mid f(x) = 1\}$ and $E_3 = \{x \mid f(x) > 1\}$.
If $x \in E_1$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} \to 0$. So by DCT, $\lim_n \int_{E_1} g_n dx = \int_{E_1} 0dx = 0$.
If $x \in E_2$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} = \frac{1}{2}$ for all $n$ and so

$$\lim_n \int_{E_2} g_n dx = \int_{E_2} \frac{1}{2} dx = \frac{1}{2} m(E_2).$$

If $x \in E_3$ then $g_n(x) = \frac{f^n(x)}{1+f^n(x)} \to 1$ and so by DCT,

$$\lim_n \int_{E_3} g_n dx = \int_{E_3} dx = m(E_3).$$

Thus,

$$\lim_n \int_0^1 g_n dx = \lim_n \int_{E_1} g_n dx + \int_{E_2} g_n dx + \int_{E_3} g_n dx = \frac{1}{2} m(E_2) + m(E_3).$$

□

**Problem 4.** *Consider $C([0,1])$ endowed with its usual uniform norm. Prove or disprove that there is a bounded linear functional $\varphi$ on $C([0,1])$ such that for all polynomials $p$, we have $\varphi(p) = p'(0)$, where $p'$ is the derivative of $p$.*

*Proof.* DISPROVE.

Consider $p_n = 1 - (x-1)^n$ so then $\|p_n\|_\infty = 1$ but $p_n'(0) = n \to \infty$. If such a $\varphi$ existed, then $n = |\varphi(p_n)| = \|\varphi(p_n)\| \leq c\|p_n\|$ which cannot happen. □

**Problem 5.** *(a) Define the product topology on the Cartesian product $\Pi_{\alpha \in A} X_\alpha$ of a family of topological spaces $(X_\alpha)_{\alpha \in A}$*

*Proof.* here! □

(b) *State Tychonoff's compactness theorem.*

*Proof.* If $\{X_\alpha\}$ is a family of compact topological spaces then $\Pi_{\alpha\in A}X_\alpha$ is compact. □

(c) *State and prove the Banach-Alaoglu theorem (Hint: Use Tychonoff's theorem)*

*Proof.* Thoerem: Let $X$ be a normed vector space. The closed unit ball $\{f \in X^* \mid \|f\| \leq 1\}$ is compact in the weak*-topology.

For all $x \in X$, let $D_x := \{\xi \mid |\xi| \leq \|x\|\} \subseteq \mathbb{C}$. Then $D_x$ is compact, and by Tychonoff's theorem, $D := \Pi_{x\in X}D_x$ is comapct. Define complex function $\varphi$ with $\varphi(x) \leq \|x\|$.

We define $B^* \subseteq D$ to consist of linear functions of $D$. We claim $B^*$ is closed. Indeed, let $\{f_\alpha\}$ be a net in $B^*$ that converges to $f$. Then

$$f(ax+by) = \lim f_\alpha(ax+by) = \lim(af_\alpha(x)+bf_\alpha(y)) = a\lim f_\alpha(x)+b\lim f_\alpha(y) = af(x)+bf(y).$$

So $f \in B^*$. Since closed subsets of comapct spaces are compact, then $B^*$ is compact in the weak*-topology. □

**Problem 6.** *Let $(X,d)$ be a compact metric space.*

(a) *Show that $X$ has a countable, dense set $\{x_n \mid n \in \mathbb{N}\}$.*

*Proof.* If $X$ is countable, we are done. So suppose $X$ is uncountable. Since $X$ is compact, for all $n \in \mathbb{N}$, $X$ can be covered by finitely many balls of radius $\frac{1}{n}$. For each $n$, choose such a finite cover with balls centered at the points $\{x_j^n\}_{j=1}^{N_n}$. Then the collection $E := \cup_n\{x_j^n\}_{j=1}^{N_n}$ is countable.

For $x \in X$, for all $n \in \mathbb{N}$, $x \in B(1/n, x_j^n)$ for some $x_j^n \in E$ so $E$ is dense. □

(b) *Let $f_n : X \to [0,\infty)$ be $f_n(x) = d(x,x_n)$. Show that if $x,y \in X$ and $f_n(x) = f_n(y)$ for all $n \in \mathbb{N}$, then $x = y$.*

*Proof.* We then have that $d(x,x_n) = d(y,x_n)$ for all $n$. We know for all $m \in \mathbb{N}$ we can find $x_m$ such that $d(x,x_m) < 1/m$ so $d(y,x_m) < 1/m$. So we can find a sequence $\{x_m\}_{m=1}^\infty$ such that $x_m \to x$ and $x_m \to y$ as $m \to \infty$. But $X$ is a metric space and thus Hausdorff, so limits are unique. Therefore, $x = y$. □

**Problem 7.** *Let $K > 0$ and let $\mathrm{Lip}_K$ be the set of functions $f : \mathbb{R} \to \mathbb{R}$ satisfying $|f(x) - f(y)| \leq K|x-y|$.*

(a) *Prove that*

$$d(f_1,f_2) = \sum_{j=0}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_1(x) - f_2(x)|$$

*defines a metric on* $\mathrm{Lip}_K$

*Proof.* First, suppose $d(f_1, f_2) = 0$. Then $\sum_{j=0}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_1(x) - f_2(x)| = 0$ so $\sup_{x\in[-j,j]} |f_1(x) - f_2(x)| = 0$ for all $j$. Thus, $f_1(x) = f_2(x)$ for all $x$.

It's trivial to see that $d(f_1, f_2) = d(f_2, f_1)$.

Finally,we'll show the triangle inequality. This again follows directly: $|f_1(x) - f_2(x)| \leq |f_1(x) - f_3(x)| + |f_3(x) - f_2(x)|$ for all $x$. Taking sup on both sodies and multiplying by $2^{-j}$ we get $d(f_1, f_2) \leq d(f_1, f_3) + d(f_1, f_2)$. □

(b) *Prove that* $\text{Lip}_K$ *is a complete metric space*

*Proof.* Suppose $(f_n)$ is a Cauchy sequence in $\text{Lip}_K$. Then for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(f_1, f_m) = \sum_{j=1}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_1(x) - f_2(x)| < \epsilon$. Then for each $j$ and $x \in [-j, j]$ we have $|f_n(x) - f_m(x)| < \epsilon'$.

Thus, $\{f_n(\xi)\}$ is Cauchy sequence on $[-j, j]$ for each $\xi$. But we can find $f(x)$ such that $f_n(x) \to f(x)$.

We want to show that $d(f_n, f) \to 0$. Since $f_n(x) \to f(x)$, then for all $\epsilon > 0$ we can find some $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - f(x)| < \epsilon$. THen

$$d(f_n, f) = \sum_{j=1}^{\infty} 2^{-j} \sup_{x\in[-j,j]} |f_n(x) - f(x)| < \sum_{j=1}^{\infty} 2^{-j}\epsilon = \epsilon$$

So $d(f_n, f) \to 0$. To see $f \in \text{Lip}_K$,

$$|f(x) - f(y)| = \left|\lim_n f_n(x) - \lim_n f_n(y)\right| = \lim_n |f_n(x) - f_n(y)| \leq K \lim_n |x - y| = K|x - y|.$$

□

**Problem 8.** *Let $X, Y$ be topological spaces. A map $f : X \to Y$ is said to be proper if for every compact subset $K \subseteq Y$, the inverse image $f^{-1}(K)$ is compact.*

(a) *Suppose $X$ is a compact space and $Y$ is Hausdorff. Prove that every continuous map $f : X \to Y$ is proper.*

*Proof.* Let $K \subseteq Y$ be compact. Since $Y$ is Hausdorff, then $K$ is closed. Since $f$ is continuous, and $Y\backslash K$ is open in $Y$ then $f^{-1}(Y\backslash K)$ is open in $X$. So $f^{-1}(K) = X\backslash f^{-1}(Y\backslash K)$ is closed. Since $X$ is compact, $f^{-1}(K)$ is compact. □

(b) *Give an example of a continuous map which is not proper.*

*Proof.* Consider the constant function $\mathbb{1} : \mathbb{R} \to \mathbb{R}$ which sends $x \mapsto 1$. So $\mathbb{1}^{-1}(\{1\}) = \mathbb{R}$. □

(c) *Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a proper continuous map. Prove that $f$ is a closed map, ie. $f(C)$ is closed in $\mathbb{R}^n$ whenever $C$ is a closed subset of $\mathbb{R}^m$.*

*Proof.* Let $\{y_n\} \subseteq f(C)$ with $y_n \to y$. Define $A = \{y\} \cup \{y_n\}$ (compact). Then $f^{-1}(A)$ is compact, so there exists $x_n \in f^{-1}(A) \cap C$ such that $f(x_n) = y_n$. Find a convergent subsequence $x_{n_k}$ with $x_{n_k} \to x$ for $x \in C \cap f^{-1}(A)$. By continuity of $f$, we have $f(x) = y$. □

**Problem 9.** *Consider the interval* $[-\pi, \pi]$ *equipped with Lebesgue measure* $\mu$. *For* $n \in \mathbb{Z}$, *consider the functions* $f_n \in C([-\pi, \pi])$ *given by* $f_n(t) = e^{int}$.

(a) *Prove that* $\text{span}_{\mathbb{C}}\{f_n \mid n \in \mathbb{Z}\}$ *is dense in the space*

$$\mathcal{A} := \{f \in C([-\pi, \pi]) \mid f(-\pi) = f(\pi)\}$$

*with respect to the uniform norm.*

*Proof.* Let $\mathcal{B} = \text{span}_{\mathbb{C}}\{f_n\} \subseteq \mathcal{A} \subseteq C([-\pi, \pi])$. Note that $\mathcal{B}$ separates points and is closed under complex conjugates. By Stone-Weierstrass, $\mathcal{B}$ is dense in $C[-\pi, \pi]$ hence also dense in $\mathcal{B}$. □

(b) *Show that* $\left\{\frac{f_n}{\sqrt{2\pi}} \mid n \in \mathbb{Z}\right\}$ *is an orthonormal basis for the Hilbert space* $L^2([-\pi, \pi], \mu)$.

*Proof.* Note that

$$\|\langle f_n, f_n\rangle\|_2 = \left|\int_{-\pi}^{\pi} e^{int} e^{-int} dt\right|^{1/2} = \sqrt{2\pi}$$

For $n \neq m$,

$$\langle f_n, f_m\rangle = \int_{-\pi}^{\pi} e^{int} e^{-int} dt = \frac{e^{i(n-m)t}}{n-m}|_{-\pi}^{\pi} = 0.$$

So they are orthonormal. □

(c) *Is the following statement true or false?:*

*"For every* $f \in \mathcal{A}$, $f = \lim_{N\to\infty} \frac{1}{2\pi} \sum_{n=-N}^{N} \langle f, f_n\rangle f_n$ *with respect to the uniform norm."*

*Give a brief explanation why or why not.*

*Proof.* TRUE.

Claim: $\sum_{-\infty}^{\infty} \langle f, f_n\rangle f_n$ exists. By Pythagorean theorem, $\left\|\sum_{-\infty}^{\infty} \langle f, f_n\rangle f_n\right\| = \sum_{-\infty}^{\infty} \|\langle f, f_n\rangle f_n\|$. By Bessel's inequality, $\sum_{-\infty}^{\infty} \langle f, f_n\rangle f_n$ is bounded so it exists.

Let $g := f - \sum_{-\infty}^{\infty} \langle f, f_n\rangle f_n$ so that

$$\langle g, f_m\rangle = \langle f, f_m\rangle - \sum_{-\infty}^{\infty} \langle\langle f, f_n\rangle f_n, f_m\rangle = \langle f, f_m\rangle - \langle f, f_m\rangle = 0.$$

By completeness of Hilbert spaces, $g = 0$. So $f = \sum_{-\infty}^{\infty} \langle f, f_n\rangle f_n$. □

**Problem 10.** *Let* $(X, \|\cdot\|)$ *be a normed linear space and let* $(X^*, \|\cdot\|_{X^*})$ *denote its dual Banach space of bounded linear functionals. Recall that* $\|\varphi\|_{X^*} = \sup_{\|x\|=1} |\varphi(x)|$ *for* $\varphi \in X^*$

(a) *Prove that for each* $x \in X$, *there exits* $\varphi \in X^*$ *with* $\|\varphi\|_{X^*} = 1$ *and* $\|x\| = \varphi(x)$.

*Proof.* We will prove the more general case: let $M$ be closed and $x \in X \backslash M$. Then there exists $\phi \in X^*$ such that $\phi(x) = \inf_{y \in M} \|x - y\|$ and $\|\phi\| = 1$ and $\phi|_M = 0$.

Restrict to the space $M + \mathbb{C}x$ and define $\phi(y + \lambda x) = \lambda \inf_{y \in M} \|x - y\|$. Then $\phi(x) = \inf_{y \in M} \|x - y\|$ and $\phi|_M = 0$.

Since $\phi(x) = \|x\|$, then $1 = \frac{\|x\|}{\|x\|} = \frac{|\phi(x)|}{\|x\|} \leq \|\phi\|$ and

$$|\phi(y + \lambda x)| \leq |\phi(y)| + |\phi(\lambda x)| = 0 + |\lambda||\phi(x)| = |\lambda| \inf_{y \in M} \|x - y\| \leq |\lambda|\|x - \lambda^{-1} y\| = \|\lambda x + y\|.$$

Therefore, $\|\phi\| = \sup_{y + \lambda x} \frac{|\phi(y+\lambda x)|}{\|\lambda x + y\|} \leq 1$ so $\|\phi\| = 1$.

Finally, if we define $p(x) = \|x\|$ for $x \in M + \mathbb{C}x$ then by Hahn-Banach, $\phi$ can be extended to $\psi$ on all $x$ with $\psi|_{M + \mathbb{C}x} = \phi$. To prove the result, set $M = \{0\}$. □

(b) *Prove that the linear map $\iota : X \to X^{**}$ given by*

$$\iota(x)(\varphi) = \varphi(x) \qquad x \in X, \varphi \in X^*$$

*is an isometry.*

*Proof.* Fix $x \in X$, so

$$\|\iota(x)\| = \frac{|\iota(x)(\phi)|}{\|\phi\|_{X^*}} = \sup_{\phi \in X^*} \frac{|\phi(x)|}{\|\phi\|_{X^*}}$$

By part (a), there exists $\phi \in X^*$ such that $\|\phi\|_{X^*} = 1$ and $\phi(x) = \|x\|$, which implies that $\|x\| \leq \|\iota(x)\|_{X^*}$.

Also, for any $\phi \in X^*$, $|\phi(x)| \leq \|\phi\|_{X^*} \|x\|$ and so

$$\|\iota(x)\| \leq \sup_{\phi \in X^*} \frac{|\phi(x)|}{\|\phi\|_{X^*}} \leq \frac{\|\phi\|\|x\|}{\|\phi\|} = \|x\|.$$

So $\|\iota(x)\| = \|x\|$ and so $\iota$ is an isometry. □

(c) *A Banach space $X$ is called reflexive if $\iota(X) = X^{**}$. Prove that the Banach space*

$$\ell^1 = \{f \in \mathbb{N} \to \mathbb{C} \mid \|f\|_1 = \sum_k |f(k)| < \infty\}.$$

*is not reflexive.*

*Hint: Consider a weak-$*$ cluster point of the sequence $(\iota(f_n))_{n \in \mathbb{N}} \subseteq (\ell^1)^{**}$, where $f_n \in \ell^2$ is the unit vector*

$$f_n(k) = \begin{cases} 1/n & k \leq n \\ 0 & k > n \end{cases}$$

*Proof.* here! □

**Problem 11.** *Let* $(g_n)_{n\in\mathbb{N}} \subseteq C([0,1])$ *be a sequence of non-negative continuous functions. Assume that for each* $k = 0, 1, 2, \ldots$ *the limit*

$$\lim_{n\to\infty} \int_0^1 x^k g_n(x)dx \qquad \textit{exists.}$$

*Prove that there exists a unique finite positive Radom measure* $\mu$ *on* $[0,1]$ *such that*

$$\int_0^1 f(x)d\mu(x) = \lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx \qquad \textit{for all } f \in C([0,1]).$$

*Proof.* Define $M := \lim_n \int_0^1 g_n(x)dx < \infty$. Let $\mathcal{A} = \operatorname{span}\{x^k \mid k \in \mathbb{N}\}$. For each $\phi \in \mathcal{A}$, by linearity, $\lim_n \int_0^1 \phi(x)g_n(x)dx$ exists.

By Stone-Weierstrass, $\mathcal{A}$ is dense in $C[0,1]$, so for every $f \in C[0,1]$, $\lim_n \int_0^1 f(x)g_n(x)dx$.

Next, let $\phi : C[0,1] \to \mathbb{C}$ be defined by $\phi(f) = \lim \int_0^1 f(x)g_n(x)dx$. Linearity is obvious. Moreover, for every $f \in C[0,1]$,

$$|\phi(f)| = \left|\lim_n \int_0^1 f g_n(x)dx\right| \le \lim_n \int_0^1 |f(x)||g_n(x)|dx \le \|f\|_n \lim_n \int_0^1 g_n(x)dx = M\|f\|_\infty$$

Hence, $\phi$ is a bounded linear functional on $C[0,1]$.

By Riesz-Representation, there exists a positive Radon measure $\mu$ such taht

$$\lim_n \int_0^1 f(x)g_n(x)dx = \phi(f) = \int_0^1 f(x)d\mu(x) \qquad \forall f \in C[0,1].$$

□

**Problem 12.** *Let* $X$ *be a locally compact Hausdorff space equipped with a Radon probability measure* $\mu$*. Let* $E \subseteq L^2(X,\mu)$ *be a closed linear subspace and assume that* $E$ *is contained in* $C_0(X)$*. The goal of this problem is to prove that* $\dim(E) < \infty$ *by justifying the following steps:*

*(a) There exists a constant* $1 \le K < \infty$ *such that*

$$\|f\|_2 \le \|f\|_u \le K\|f\|_2 \quad \textit{for all } f \in E,$$

*where* $\|\cdot\|_u$ *denotes the uniform norm. Hint: us the closed graph theorem for one of the inequalities.*

*(b) For each* $x \in X$*, there exists a unique* $g_x \in E$ *such that* $\|g_x\|_2 \le K$ *and*

$$f(x) = \langle f, g_x\rangle \qquad \textit{for all } f \in E.$$

*(c) Let* $(f_i)_{i\in I}$ *be any orthonormal basis for* $E$*. Then*

$$\sum_{i\in I} |f_i(x)|^2 = \|g_x\|_2^2 \leq K^2 \qquad \textit{for all } x \in X.$$

*(d)* $\dim(E) = |I| \leq K^2$.

*Proof.* See January 2017, Problem #5 for a solution to a similar question. □

## 4 January 2018

**Problem 1.** *Suppose* $U_1, U_2, \ldots$ *are open subsets of* $[0,1]$*. In each case, either prove the statement or disprove it.*

*(a) If* $\lambda\big(\cap_{n=1}^\infty U_n\big) = 0$ *then for some* $n \geq 1$*, we have* $\lambda(\overline{U_n}) < 1$*, where* $\lambda$ *is Lebesgue measure and* $\overline{U_n}$ *is the closure of* $U_n$ *in the usual topology on* $[0,1]$*.*

*Proof.* FALSE. Let $r_m$ be an enumeration of the rationals on $[0,1]$, and set $a_{n,m} = 1/2^{n+m}$. Set

$$U_n := \cup_m (r_m - a_{n,m}, r_m + a_{n,m})$$

These are open since they are a union of open intervals. Moreover, since $\mathbb{Q} \subseteq U_n$ then $\lambda(\overline{U_n}) = \lambda([0,1]) = 1$. But by upper continuity of the Lebesgue measure, then

$$\lambda\left(\cap U_n\right) = \lim_m \lambda\left(\cup(r_n - a_{n,m}, r_n + a_{n,m})\right) = 0.$$

□

*(b) If* $\cap_{n=1}^\infty U_n = \emptyset$*, then for some* $n \geq 1$*, the set* $[0,1]\backslash U_n$ *contains a non-empty open interval.*

*Proof.* TRUE. Recall that the Baire Category Theorem states that under these assumptions, if each $U_n$ is dense in $[0,1]$ then $\cap_{n=1}^\infty U_n$ is also dense in $[0,1]$. Then since we have that $\cap_{n=1}^\infty U_n$ is not dense, then there must be some $n$ such that $[0,1]\backslash U_n$ is not dense. This precisely means that $U_n$ contains a non-empty open interval. □

**Problem 2.** *Let* $X$ *be a separable compact metric space and show that* $\mathcal{C}(X)$ *is separable.*

*Proof. Remark:* If $X$ is a compact metric space, then $X$ is separable. So the separable assumption is superfluous.

Suppose $d$ is the metric on $X$ and $(x_n)$ is a dense countable subset of $X$. For each $n \in \mathbb{N}$, define the functional $f_n$ by $f_n(x) := d(x, x_n)$. Then each $f_n$ is a continuous functional. Consider $F = \{1, f_1, f_2, \ldots\}$ and consider the subalgebra generated by the rational span of $F$, call it $\mathbb{Q}[F]$ (this is still countable, we can consider the span, then cosider the set where two elements of it are multiplied together, then the set where three elements are multiplied together, etc). This is countable and dense in $\mathcal{A} := \mathbb{R}[F]$. so it is sufficient to show that $\mathcal{A}$ is dense in $\mathcal{C}(X)$.

We will attempt to use the Stone Weierstrass Theorem:

By definition, $\mathbb{R}[F]$ contains the constant function 1. We are left to show it separates points. Take two points $x \neq y$ in $X$. Since $\{x_n\}$ is dense, then there must exist some $m$ such that $d(x, x_m) \leq \frac{1}{3}d(x, y) \neq 0$. If $d(y, x_m) = d(x, x_m)$ then

$$d(x, y) \leq d(x, x_m) + d(y, x_m) = 2d(x, x_m) \leq \frac{2}{3}d(x, y).$$

This cannot be true under our assumption $d(x, y) \neq 0$. So then $f_m(y) = d(y, x_m) \neq d(x, x_m) = f_m(x)$. So $f_m$ separates $x$ and $y$.

Therefore, by Stone-Weierstrass, $\mathcal{A}$ is dense in $\mathcal{C}(X)$. But $\mathcal{Q}[F]$ is countable and dense in $\mathcal{A}$, so therefore $\mathcal{C}(X)$ is separable. □

**Problem 3.** *Let $f : [0, 1] \to \mathbb{R}$ be a bounded Lebesgue measurable function such that*

$$\int_0^1 f(t)e^{nt}dt = 0$$

*for every $n \in \{0, 1, 2, \ldots\}$. Prove that $f(t) = 0$ for almost every $t \in [0, 1]$.*

*Proof.* Let $f(t) = 0$ on $t = 0, 1$. Using Stone-Weierstrass to show we can pass to the case $\int_0^1 f(t)g(t) = 0$ for all $g \in \mathcal{C}[0, 1]$.

By a standard density argument, we may pass to the case where $g$ is a step function. We claim that $f = 0$ a.e.

Assume not. WLOG there exists some $E = \{x \in [0, 1] \mid f(x) > 0\}$ with $m(E) > 0$ (else consider $-f$).

Since $f$ is bounded, then $E_\infty := \{x \in [1, 2] \mid f(x) = \infty\}$ is a null set. Define $E_n := \{x \in [0, 1] \mid 1/n < f(x) < n\}$. We can write $E = (\bigcup_n E_n) \cup E_\infty$. So there exists some $N$ such that $m(E_N) = a > 0$.

We can write $A$ as a finite disjoint union of open intervals, $A = \sqcup_{i=1}^m I_i$, such that $m(E_N \triangle A) < \epsilon$ and $A \subseteq E_N$.

Put $g = \sum_{i=1}^m \chi_{I_i}$, then $\int_1^2 g(x)f(x) = \int_{E_N} f(x)dx$. Since

$$\left|\int_{E_N} f(x) - \int_A f(x)\right| \leq Nm(E_N \triangle A) < N\epsilon$$

If we choose $\epsilon$ small enough, we see the contradiction since $\int_0^1 g(x)f(x) > 0$. □

**Problem 4.** *(a) Prove that every compact subset of a Hausdorff space is closed.*

*Proof.* Let $A$ be a compact subset of the Hausdorff space $X$. To show $A$ is closed, we'll show $A^c = X \backslash A$ is open. Take $x \in X \backslash A$. Then for every $y \in A$, there are disjoint sets $U_y$ and $V_y$ with $x \in V_y$ and $y \in U_y$.

The collection of open sets $\{U_y \mid y \in A\}$ forms an open cover of $A$. Since $A$ is compact, this open cover has a finite subcover, $U_{y_1}, U_{y_2}, \ldots, U_{y_n}$. Let

$$U := \cup_{i=1}^n U_{y_i} \qquad V := \cap_{i=1}^n V_{y_i}$$

Since each $U_{y_i}$ and $V_{y_i}$ are disjoint, then $U$ and $V$ are disjoint. Also, $A \subseteq U$ and $x \in V$.

Thus, for every point $x \in X \backslash A$ we have found an open set $V$ containing $x$ which is disjoint from $A$. So $X \backslash A$ is open and $A$ is closed. □

(b) *Let $f : X \to Y$ be a bijective continuous function between topological spaces. Suppose that $X$ is compact and $Y$ is Hausdorff and prove that $f$ is a homeomorphism.*

*Proof.* Let $g = f^{-1}$. We need to show that $g$ is continuous.

For every $V \subseteq X$, we have $g^{-1}(V) = f(V)$. We want to show that if $V$ is closed in $X$ then $g^{-1}(V)$ is closed in $Y$.

Suppose $V$ is closed in $X$. Since $X$ is compact, $V$ is compact by part (a). So $f(V)$ is compact since the continuous image of a compact space is compact.

Since $Y$ is Hausdorff, $f(V)$ is closed by the fact that a compact subspace of Hausdorff space is closed. But $f(V) = g^{-1}(V)$ so $g^{-1}(V)$ is closed. So $g$ is continuous and $f$ is a homeomorphism. □

(c) *Prove or disprove that if $X$ is a dense subset of a topological space $Y$ and if $X$ is Hausdorff in the relative topology, then $Y$ is also Hausdorff.*

*Proof.* FALSE. Consider $Y = \{a, b\}$ with discrete topology $\tau = \{\emptyset, \{a, b\}\}$. Let $X = \{a\}$ with relative topology $\tau_X = \{\emptyset, \{a\}\}$.

Then it's easy to see $X$ is dense in $Y$ (since every open set containing an element of $Y$ has nonempty intersection with $X$, trivially). Since $X$ has only a single element, it's Hausdorff in the relative topology trivially. But $Y$ is not Hausdorff. □

**Problem 5.** *Prove that the following limit exists and compute its value:*

$$\lim_{n\to\infty} \int_0^n \left( \sum_{k=0}^n \frac{(-1)^k x^{2k}}{(2k)!} \right) e^{-2x} dx.$$

*Proof. Solution from Sheagan John*

Let us first note an important simplification of the integrand, by considering the Taylor series expansion of $\cos x$ around a neighbourhood of 0.

$$\cos x = \sum_{k=0}^\infty \frac{(-1)^k}{(2k)!} x^{2k}$$

Letting $f(x) := (\cos x)e^{-2x}$ and $f_k(x) := e^{-2x} \frac{(-1)^k}{(2k)!} x^{2k}$ then it's clear that $\{f_k\}$ is a convergent sequence which converges to $f(x)$.

Now, since $|\cos x| \leq 1$ we further have that for some positive constant $c < 2$

$$|f_n(x)| \leq c|f(x)| = c|\cos x e^{-2x}| = c|\cos x|e^{-2x} \leq ce^{-2x}$$

Since $g(x) = ce^{-2x}$ is integrable on the positive half line

$$\int_0^\infty ce^{-2x}\,dx = \frac{-c}{2}e^{-2x}\Big|_0^\infty = \frac{c}{2}$$

the dominated convergence theorem can be applied to the original Lebesgue integral limit, with $f_n \longrightarrow f$, and with $g$ as the dominating function.

$$\lim_{n\longrightarrow\infty}\int_0^n \left(\sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}x^{2k}\right)e^{-2x}\,dx = \lim_{n\longrightarrow\infty}\int_0^n f_n(x)\,dx = \int_0^\infty f(x)\,dx = \int_0^\infty e^{-2x}\cos x\,dx$$

Recall that the Laplace transform of $\cos x$ is $\int_0^\infty e^{-st}\cos at dt = \mathcal{L}\{\cos ax\}(s) = \frac{s}{s^2+a^2}$. Therefore, the last integral is equal to $\frac{2}{2^2+1} = 2/5$. □

**Problem 6.** *Let $X$ and $Y$ be Banach spaces (over $\mathbb{C}$)*

*(a) A linear map $T : X \to Y$ is called adjointable if $T^*f \in X^*$ for every $f \in Y^*$. Prove that $T$ is adjointable if and only if $T \in \mathcal{B}(X,Y)$.*

*Proof.* $\Leftarrow$) if $T \in \mathcal{B}(X,Y)$ then by definition, for every $f \in Y^*$, we have $T^*f \in X^*$

$\Rightarrow$) Suppose $T^*f \in X^*$ for every $f \in Y^*$. We will use the Closed Graph Theorem. Suppose $x_n \to x$ in $X$ and that $Tx_n \to y$ in $Y$. Then since $T^*f \in X^*$ for every $f \in Y^*$ we can apply this to the convergence to see that

$$f(Tx_n) = (T^*f)(x_n) \to (T^*f)(x) = f(Tx) \qquad \forall f \in Y^*$$

By the Hahn-Banach theorem, $Y^*$ separates points in $Y$ so therefore, $Tx_n \to Tx$. Uniqueness of limits implies $Tx = y$ and so the graph of $T$ is closed. By the Closed Graph Theorem, $T$ is bounded. □

*(b) Suppose a bounded linear functional $\psi : X^* \to \mathbb{C}$ is weak*- continuous. Show (from the definitions) that there exists $x \in X$ such that $\psi(\phi) = \phi(x)$.*

*Proof.* Define the functional

$$\begin{aligned} \mathrm{ev}_x : X^* &\to \mathbb{C} \\ f &\mapsto f(x) \end{aligned}$$

We want to show that every bounded, linear, weak*-continuous functional $\psi : X^* \to \mathbb{C}$ is of this form.

Indeed, since $\psi$ is weak*-continuous, then it is weak* continuous at 0. Thus, the set $\{f \in X^* \mid |\psi(f)| < 1\}$ is weak* open and must containa neighborhood of 0. By definition of weak* topology, there must exist $x_1, \ldots, x_n \in X$ such that

$$V(x_1, \ldots, x_n) := \{f \in X^* \mid |f(x_i)| \leq 1, i = 1, \ldots, n\} \subseteq \{f \in X^* \mid |\psi(f)| < 1\}.$$

Then we will next show that $\cap_{i=1}^n \ker(\mathrm{ev}_{x_i}) \subseteq \ker(\psi)$.

Indeed, let $f \in \ker(\mathrm{ev}_{x_i})$ so $|f(x_i)| = 0$ for all $i = 1, \ldots, n$. Take $\epsilon > 0$ and conser $g = \frac{1}{\epsilon} f$, so $|g(x_i)| = \frac{1}{\epsilon}|f(x_i)| = 0$ for all $i = 1, \ldots, n$. In particular, $g \in V(x_1, \ldots, x_n)$ and so then we have if $|\psi(g)| < 1$ then $|\psi(f)| < \epsilon$. But $\epsilon$ is arbitrary so $\psi(f) = 0$, i.e. $f \in \ker(\psi)$.

Now recall the linear algebra trick that says if for linear functionals $\ker(T) \subseteq \ker(S)$ then $S$ is a scalar multiple of $T$. In this case, we get that $\psi$ is a linear combination of the $\mathrm{ev}_{x_i}$, i.e. is of the form $\mathrm{ev}_x$ where $x$ is a linear combination of the $x_i$'s.

Moreover, because the weak* topology is Hausdorff, $x$ is necessarily unique. □

(c) *Let $S \in \mathcal{B}(Y^*, X^*)$. Prove that $S$ is weak*-weak*-continuous if and only if $S = T^*$ for some $T \in \mathcal{B}(X, Y)$.*

*Proof.* $\Leftarrow$) If $S = T^*$ then if $f_\alpha \to f$ is a weak* convergent net in $Y^*$ then for any $y \in Y$, $f_\alpha(y) \to f(y)$. Therefore,

$$\underbrace{(Sf_\alpha - Sf)}_{\in X^*}(x) = (Tx)\underbrace{(f_\alpha - f)}_{\to 0} \to 0.$$

So $S$ is weak*-weak* continuous.

$\Rightarrow$) Suppose $S : Y^* \to X^*$ is weak*-weak* continuous. Then the evaluation function on $x$, $\mathrm{ev}_x(S)$ is weak* continuous on $Y^*$ (where $\mathrm{ev}_x(S) : Y^* \to \mathbb{C}, (\mathrm{ev}_x(S))(f) = (Sf)(x)$).

By part (b), we know that $\mathrm{ev}_x(S)$ is of the form $\mathrm{ev}_{T(x)}$ for some unique $T(x) \in Y$. Since $T(x)$ is uniquely determined, it follows that $T$ is linear.

We will now check that $T$ is continuous by the closed graph theorem: if $x_n \to x$ adn $Tx_n \to y$ in norm then for each $\phi \in Y^*$ we have

$$\langle \phi, y \rangle = \lim \langle \phi, Tx_n \rangle = \lim \langle S\phi, x_n \rangle = \langle S\phi, x \rangle = \langle \phi, Tx \rangle.$$

And so $y = Tx$ as desired. So $T$ is bounded and therefore, $S = T^*$ is bounded as well. □

**Problem 7.** *Let $(f_n)_{n=1}^\infty$ be a sequence of functions $f_n : [0, 1] \to \mathbb{R}$.*

*NOTE: I think we also require continuous....*

(a) *What does it mean for $\{f_n \mid n \geq 1\}$ to be equicontinuous?*

*Proof.* $\{f_n \mid n \geq 1\}$ is said to be equicontinuous if for every $\epsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in [0, 1]$, if $|x - y| < \delta$ then $|f(x) - f(y)| < \epsilon$. □

(b) *Suppose that for every $n$, $f_n$ is differentiable and $|f_n'(t)| \leq 1$ for all $t$. Prove that $\{f_n \mid n \geq 1\}$ is equicontinuous.*

*Proof.* Since $|f_n'(t)| \leq 1$ for all $t$, then for all $n$, we have by the mean value theorem that

$$\frac{|f_n(x) - f_n(y)|}{|x-y|} \leq 1.$$

Hence, for any fixed $\epsilon > 0$, setting $\delta = \epsilon$ and for $|x - y| < \delta$ then

$$|f_n(x) - f_n(y)| \leq |x - y| < \delta = \epsilon.$$

□

(c) *Suppose the hypothesis of (b) holds and assume in addition that $|f_n(0)| \leq 1$ for every $n \geq 1$. Prove that there exists a continuous function $f : [0, 1] \to \mathbb{R}$ and a subsequence $(f_{n(k)})_{k=1}^{\infty}$ converging uniformly to $f$.*

*Proof.* This is essentially the Arzela-Ascoli Theorem. Since $|f_n(0)| \leq 1$ for all $n$ and since $|f_n'(t)| \leq 1$ for all $t$, then $|f_n(t)| \leq 2$ for all $t \in [0, 1]$ and for all $n$. That is, $\{f_n\}$ is uniformly bounded. It's also equicontinuous by part (b). Therefore, Arzela-Ascoli theorem states that there is a subsequence $\{f_{n_k}\}$ which converges uniformly. Let $f$ be the limit, and we finish by recalling that the uniform convergence of continuous functions is also continuous.

*Note:* it might be good to know the Arzela-Ascoli Theorem. □

(d) *Show by example that the limit function $f$ need not be differentiable.*

*Proof.* Take $f_n(x) = \sqrt{x^2 + 1/n}$ so $f_n(0) = \frac{1}{\sqrt{n}} \leq 1$ for all $n$ and so $\{f_n\}$ is uniformly bounded. Next, we can see that $f_n'(x) = \frac{x}{\sqrt{x^2+1/n}}$ so that for $x \in [0, 1]$ we have $|f_n'(x)| \leq \sqrt{\frac{n}{n+1}} \leq 1$ as desired.

However, it's also clear that the limit must be $f = |x|$ which is not differentiable. □

**Problem 8.** *Let $\mathcal{H}$ be a complex Hilbert space. Given a non-empty set $E \subseteq \mathcal{H}$ and $x \in \mathcal{H}$, put* $\operatorname{dist}(x, E) = \inf\{\|x - y\| \mid y \in E\}$ *and* $E^\perp = \{x \in \mathcal{H} \mid \langle x, y\rangle = 0\ \forall y \in E\}$.

(a) *Let $\mathcal{H}_0 \subseteq \mathcal{H}$ be a closed subspace and $x \in \mathcal{H}$. Prove that there exists $x_0 \in \mathcal{H}_0$ such that $\|x - x_0\| = \operatorname{dist}(x, \mathcal{H}_0)$.*

*Proof.* Let $\delta = \operatorname{dist}(x, \mathcal{H}_0)$. Then there exists a sequence $(y_n) \in \mathcal{H}_0$ such that $\delta_n := \|x - y_n\| \to \delta$. We will show that $(y_n)$ is Cauchy. Indeed,

$$0 \leq \|y_n - y_m\|^2 = -\|y_n + y_m - 2x\|^2 + 2(\|y_n - x\|^2 + \|y_m - x\|^2) \leq -4\delta^2 + 2(\delta_n^2 + \delta_m^2) \to 0.$$

where we use the fact that

$$\|y_n + y_m - 2x\|^2 = 4\left\|\underbrace{\frac{y_n + y_m}{2}}_{\in \mathcal{H}_0} - x\right\|^2 \leq 4\delta.$$

Thus, $(y_n)$ is a Cauchy sequence and so because we are in a Hilbert space, $(y_n)$ converges to some point $x_0 \in \mathcal{H}$. Since $\mathcal{H}_0$ is closed and $y_n \in \mathcal{H}_0$ for all $n$ then we get that $x_0 \in \mathcal{H}_0$. Finally, $\|x - x_0\| = \lim \|x - y_n\| = \lim \delta_n = \delta$.

*Exercise:* it can be shown if $\mathcal{H}_0$ is convex, then the choice of $x_0$ is unique! □

(b) *With $x$ and $x_0$ as above, prove that $x - x_0$ is orthogonal to $\mathcal{H}_0$.*

*Proof.* Let $y \in \mathcal{H}_0$ be an arbitrary vector with $\|y\| = 1$, set $\alpha := \langle x - x_0, y\rangle$. Then since $\overline{\alpha}\langle x - x_0, y\rangle = \overline{\alpha}\alpha = |\alpha|^2$ and $\alpha\langle y, x - x_0\rangle = \alpha\overline{\alpha} = |\alpha|^2$, we have

$$\|x-(x_0+\alpha y)\|^2 = \|x-x_0-\alpha y\|^2 = \|x-x_0\|^2 - \overline{\alpha}\langle x-x_0, y\rangle - \alpha\langle y, x-x_0\rangle + |\alpha|^2 = \|x-x_0\|^2 - |\alpha|^2.$$

So since $x_0 + \alpha y \in \mathcal{H}_0$ then $\|x - x_0 - \alpha y\| \geq \|x - x_0\|$. Hence $\alpha = 0$.

Therefore, for any nonzero $y \in \mathcal{H}_0$ we can write

$$\langle x - x_0, y\rangle = \|y\|\langle x - x_0, y/\|y\|\rangle = \|y\|0 = 0.$$

So $\langle x - x_0, y\rangle = 0$ for all $y \in \mathcal{H}_0$ so $x - x_0 \perp \mathcal{H}_0$. □

(c) *Prove that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ (the algebraic direct sum)*

*Proof.* This follows immediately from parts (a) and (b). Take some arbitrary $x \in \mathcal{H}$. We can find the appropriate $x_0$ as above, so $x = x_0 + (x - x_0) \in \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$.

The fact that it is a direct sum follows from the fact that $\mathcal{H}_0 \cap \mathcal{H}_0^\perp = \{0\}$. □

(d) *Let $E \subseteq \mathcal{H}$ be non-empty. Prove that $(E^\perp)^\perp = E$ if and only if $E$ is a closed subspace.*

*Proof.* If $E$ is closed, then the above parts (a),(b), and (c) apply and prove that $(E^perp)^\perp = E$. To see the converse, we will instead show that $(E^\perp)^\perp = \overline{E}$. The desired result will then immediately follow.

Since $E \subseteq \overline{E}$ then $\overline{E}^\perp \subseteq E^\perp$ and therefore, $(E^\perp)^\perp \subseteq (\overline{E}^\perp)^\perp$. Since $\overline{E}$ is closed, then $(\overline{E}^\perp)^\perp = \overline{E}$ so $(E^\perp)^\perp \subseteq \overline{E}$.

Conversely, since $E^\perp$ is closed for every $E$ (independent of whether $E$ is closed or not) then $(E^\perp)^\perp$ is closed and so since $E \subseteq (E^\perp)^\perp$, then by the monotonicity of topological closure we have that $\overline{E} \subseteq \overline{(E^\perp)^\perp} = (E^\perp)^\perp$.

Therefore, $(E^\perp)^\perp = \overline{E}$. □

**Problem 9.** *Let $V$ be a vector space over $\mathbb{R}$ or $\mathbb{C}$. Recall that a Hamel basis for $V$ is a linearly independent subset of $V$ whose linear span equals $V$.*

*(a) Let $S \subseteq V$ and suppose the linear span of $S$ equals $V$. Show that $V$ has a Hamel basis that is a subset of $S$.*

*Proof.* Choose a Hamel basis $B$ of $S$. Then it is easy to check that $B$ is a Hamel basis of $V$. □

*(b) Suppose $V$ has an infinite Hamel basis and show that all hamel bases of $V$ have the same cardinality.*

*Proof.* Suppose that $\{v_i\}_{i\in I}$ and $\{u_j\}_{j\in J}$ are two infinite bases for $V$. For each $i \in I$, then $v_i$ is in the linear span of $\{u_j\}_{j\in J}$. Therefore, there exists a finite subset $J_i \subseteq J$ such that $v_i$ is in the linear span of the vectors $\{u_j\}_{j\in J_i}$. Therefore, $V = \text{span}(\{v_i\}_{i\in I}) \subseteq \text{span}\{u_j\}_{j\in \cup J_i}$. Since no proper subset of $\{u_j\}_{j\in J}$ can span $V$, it follows that $J = \cup_{i\in I} J_i$. Therefore $|J| \leq |I|$.

A symmetric argument shows that $|I| \leq |J|$. □

**Problem 10.** *Suppose $(X, \mathcal{M}, \rho)$ is a finite measure space and $\mathcal{A} \subseteq \mathcal{M}$ is an algebra of sets with a finitely additive complex measure $\mu : \mathcal{A} \to \mathbb{C}$ such that $|\mu(E)| \leq \rho(E)$ for all $E \in \mathcal{A}$. Show that there exists a complex measure $\nu : \mathcal{M} \to \mathbb{C}$ whose restriction to $\mathcal{A}$ is $\mu$ and such that $|\nu(E)| \leq \rho(E)$ for all $E \in \mathcal{M}$.*

*Hint: you may want to consider the subspace $V \subseteq L^1(\rho)$ that is spanned by the set of characteristic functions $\chi_E$ for $E \in \mathcal{A}$, and a certain linear functional on $V$.*

*Proof. Solution from Minh Kha.*

For each subalgebra $\mathcal{U}$ of $\mathcal{M}$, we define $S_{\mathcal{U}}$ to be the set of all simple functions of the form $\sum_{i=1}^n c_i\chi_{E_i}$ where $c_i \in \mathbb{R}$, $E_i \in \mathcal{U}$. Then $S_{\mathcal{A}}$ is a vector subspace of $S_{\mathcal{M}}$.

Now define $p : S_{\mathcal{M}} \to \mathbb{R}$ such that

$$p(f) = \sup\left\{\sum_{i=1}^n |c_i|\rho(E_i) \;\middle|\; f = \sum_{i=1}^n c_i\chi_{E_i}, E_i \cap E_j = \emptyset\ \forall i \neq j, E_i \in \mathcal{M}, c_i \in \mathbb{R}\right\} \quad \forall f \in S_{\mathcal{M}}$$

It's not difficult to check that $p$ satisfies $p(f+g) \leq p(f) + p(g)$ for all $f, g \in S_{\mathcal{M}}$ and $p(tf) = tp(f)$ for all $t \in \mathbb{R}^+, f \in S_{\mathcal{M}}$. Thus, $p$ is a seminorm and is just an extension of the total variation of the measure $p$ when you apply to the function $f = 1$.

Define a linear map $T : S_{\mathcal{A}} \to \mathbb{R}$ defined by

$$T(f) = \int_X f d\mu \quad \forall f \in S_{\mathcal{A}}$$

This is linear because of the finite additive property of $\mu$. Then $|T(f)| \leq p(f)$ for all $f \in S_{\mathcal{A}}$. By Hahn-Banach, we get a linear extension of $T$ on $S_{\mathcal{M}}$, which we denote by $\widetilde{T}$. Moreover, this extension $\widetilde{T} : S_{\mathcal{M}} \to \mathbb{R}$ satisfies $|\widetilde{T}(f)| \leq p(f)$ for all $f \in S_{\mathcal{M}}$.

Now, we define a finite additive measure $\nu$ on $\mathcal{M}$ by letting $\nu(E) = \widetilde{T}(\chi_E)$ for all $E \in \mathcal{M}$. Thus, $\nu|_{\mathcal{A}} = \mu$ and $|\nu(E)| \leq p(E)$ for all $E \in \mathcal{M}$.

To check the countably additive property of $\nu$, consider any countable collection of disjoint measurable subsets $E_i \in \mathcal{M}$ and so $\chi_{\cup_i E_i} = \sum_i \chi_{E_i}$. Thus, $\nu(\cup_i E_i) = \sum_i \nu(E_i)$ since the series $\sum_i \widetilde{T}(E_i)$ converges (use $|\widetilde{T}(f)| \leq p(f)$ for all $f \in S_{\mathcal{M}}$ and properties of the measure $p$).

For the complex case, repeat the trick by proving the complex version of the Hahn-Banach theorem from the real version. □

# 5 August 2017

**Problem 1.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space and let $\{f_n\}$ be a sequence of measurable functions on $X$. Prove, directly from the definition of convergence almost everywhere, that if $\sum_n \mu[|f_n| > 1/n] < \infty$, then the sequence $\{f_n\}$ converges almost everywhere to zero. Deduce that every sequence of measurable functions that converges in measure to zero has a subsequence that converges almost everywhere to zero.*

*Proof.* Let $E = \{x \in \Omega \mid \lim_n |f_n(x)| = 0\}$. We want $\mu(E^c) = 0$. Let

$$M = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > 1/n\}$$

Since

$$\mu\left(\bigcup_{n=m}^{\infty} \left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) \leq \sum_{n=m}^{\infty} \mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) \quad \to \quad 0.$$

Therefore, $\mu(M) = 0$ and

$$M^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| \leq 1/n\}.$$

Note: $f_n(x) \to 0$ if and only if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n > N$, $|f_n(x)| < \epsilon$.

So for any $x \in M^c$ choose $1/N < \epsilon$ s.t. $\forall n > N$ we have $|f_n(x)| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$.

Therefore $M^c \subseteq E$, so $E^c \subseteq M$, implying $\mu(E^c) = 0$. So then $\{f_n\}$ converges almost everywhere to zero.

Step 2: We will show that if $f_n \to 0$ in measure, then there exists a subsequence that converges to 0 pointwise almost everywhere.

Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{f_{n_k}\}$ such that if

$$E_j = \{x \mid |f_{n_j}(x) - f_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \cup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \cap_k F_k$ so $\mu(F) = 0$. For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |f_{n_\ell}(x) - f_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So $f_{n_k}$ is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim f_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $f_{n_k} \to 0$ almost everywhere and $f_n \to f$ in measure since

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f_n(x) - f_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |f_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - f_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |f_n(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = 0$ almost everywhere. Thus, $\{f_{n_k}\}$ converges to 0 almost everywhere. □

**Problem 2.** *Show that there is a sequence of nonnegative functions $\{f_n\}$ in $L^1(\mathbb{R})$ such that $\|f_n\|_{L^1(\mathbb{R})} \to 0$, but for any $x \in \mathbb{R}$, $\limsup_n f_n(x) = \infty$.*

*Proof.* We will explicitely construct such a sequence. Consider the following pattern:

To cover $[-1, 1]$ let

$$f_1 = \sqrt{1}\chi_{[-1,0]}, \qquad f_2 = \sqrt{1}\chi_{[0,1]}.$$

so that $\|f_1\|_{L^1(\mathbb{R})} = 1 = \|f_2\|_{L^1(\mathbb{R})}$. To cover $[-2, 2]$, next let

$$f_3 = \sqrt{2}\chi_{[-2,-1.5]}, \qquad f_4 = \sqrt{2}\chi_{[-1.5-1]} \qquad \ldots, f_{10} = \sqrt{2}\chi_{[1.5,2]}$$

so then $\frac{1}{\sqrt{2}} = \|f_3\|_{L^1(\mathbb{R})} = \|f_4\|_{L^1(\mathbb{R})} = \ldots = \|f_10\|_{L^1(\mathbb{R})}$. Next, we cover $[-3, 3]$ so that

$$f_{11} = \sqrt{3}\chi_{[-3,-2.666]}, \ldots f_{28} = \sqrt{3}\chi_{[2.666,3]}$$

so that $\frac{1}{\sqrt{3}} = \|f_{11}\|_{L^1(\mathbb{R})} = \ldots = \|f_{28}\|_{L^1(\mathbb{R})}$. If we continue in this fashion, we get the desired functions.

Explicitely, for $n = \sum_{i=1}^{N-1} 2i^2 + k = \frac{1}{3}(N-1)(N)(2N-1) + k$ where $N \in \mathbb{N}$, $0 \leq k < 2N^2$, then we set

$$f_n = \sqrt{N}\chi_{[-N+k/N, -N+(k+1)/N]}$$

so that $\|f_n\|_{L^1(\mathbb{R})} = \frac{1}{\sqrt{N}}$ but for every $x \in \mathbb{R}$, it's clear that $\limsup_n f_n(x) = \infty$. □

**Problem 3.** *Construct a sequence of nonnegative Lebesgue measurable functions $\{f_n\}$ on $[0,1]$ such that*

*(a) $f_n \to 0$ almost everywhere, and*

*(b) for any interval $[a,b] \subseteq [0,1]$,*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = b - a.$$

*Proof.* **Claim.** For any $f \geq 0$ that is continuous on $[0,1]$, $n\int_{[x,x+\frac{1}{n^2}]}(f(y) - f(x))dy = o(\frac{1}{n})$.

For any $\epsilon > 0$, $\exists N \in \mathcal{N}$ such that $0 < x < \frac{1}{N}$ then $f(x) - f(0) < \varepsilon$. Hence for $n > N$, $n\int_{[0,\frac{1}{n^2}]}(f(x) - f(0)) < n\epsilon\frac{1}{n^2}$.

A clear extension to this is that $\int_{[x,x+\frac{1}{n}]}(f(y) - f(x))dy = o(\frac{1}{n})$.

Let $f_n(x) := \sum_{k=0}^{n-1} n\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]}$, and let $\varepsilon, N$ be as above. Clearly $f_n$ is measurable and satisfies (a) above. To prove (b), observe that, for $n > N$,

$$\begin{aligned} I_k &:= \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} nf\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]} - f \right| \\ &= \left| n\int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} f - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f \right| \\ &= o\left(\frac{1}{n}\right). \end{aligned}$$

Hence $|\int(ff_n - f)| \leq \sum_{k=0}^{n-1} I_k = o(1)$.

This holds for all $f \in \mathcal{C}([0,1])$, so in particular, it will hold for $f = \chi_{[a,b]}$. □

**Problem 4.** *In this problem the measure is Lebesgue measure on $[0,1]$. The norm on $L^\infty[0,1]$ is the essential supremum norm, which for a continuous function is the same as the supremum norm.*

*(a) Prove or disprove that $L^\infty[0,1]$ is separable in the norm topology.*

*Proof.* $L^\infty[0,1]$ is not separable in the norm topology. Consider the collection of functions $f_r = \chi_{[-r,r]}$ for real $1 \geq r > 0$. Since there are uncountably many such $r$ and since $\|f_r - f_{r'}\|_\infty = 1$ for any $r \neq r'$, it's impossible to have a countable subset of $L^\infty[0,1]$ that is dense in it. □

*(b) Recall that $L^\infty[0,1] = (L^1[0,1])^*$. What is the weak* closure in $L^\infty[0,1]$ of the unit ball of $C[0,1]$? Prove your assertion.*

*Proof.* <u>More general claim:</u> For $X$ an infinite dimensional Banach space, $\overline{S_{X^*}}^{w^*} = B_{X^*}$.

We know for any $x_1, x_2, \ldots, x_n \in X$, there exists some $x_0^* \neq 0$ such that $x_0^*(x_i) = 0$. Indeed, if this were not true then otherwise, $x_0^*(x_i) \neq 0$ for some $i$, let $\varphi : X^* \to \mathbb{R}^n$ be $\varphi(x^*) = (x^*(x_1), \ldots, x^*(x_n))$ then $\varphi$ is injective so $\dim(X^*) \leq \dim(\mathbb{R}^n) = n$. Contradiction, so true.

Now for any $x^* \in B_{X^*}$, consider it's neighborhood (open under the $w^*$-neighborhood)

$$V = \cap_{i=1}^n \{y^* \in X^* \mid |\hat{x_i}(x^* - y^*)| = |x^*(x_i) - y^*(x_i)| < \epsilon\}$$

for each $\{x_i\}_{i=1}^n$ choose such an $x_0^* \neq 0$ from the claim.

Consider the line $\{x^* + tx_0^* \mid t \in \mathbb{R}\}$ in $X^*$.

Since for any $\hat{x_i}$,

$$\hat{x_i}(x^* + tx_0^* - x^*) = t\hat{x_i}(x_0^*) = tx_0^*(x_i) = 0 < \epsilon.$$

Then $\{x^* + tx_0^* \mid t \in \mathbb{R}\} \subseteq V$. Since $\|x^* + tx_0^*\|$ is continuous about $t$, then we can find $t_0$ such that $\|x^* + t_0 x_0^*\| = 1 \Rightarrow V \cap S_{X^*} \neq \emptyset$.

Since any neighborhood of $x^*$ contains a neighborhood of the form $V$ as above (i.e. these $V$'s are a neighborhood basis) then $B_{X^*} \subseteq \overline{S_{X^*}}^{w^*}$.

On the other hand, for any $x_0^* \in B_{X^*}$, by Hahn-Banach separation Theorem, we know there exists $x \in X$ and $c \in \mathbb{R}$ such that $x^*(x) < c < x_0^*(x)$ for all $x^* \in B_{X^*}$.

Then for all $\{x_n^*\} \subseteq B_{X^*}$, $x_n^*(x) \leq c < x_0^*(x)$. Therefore, $x_0^*$ isn't an accumulation point of $B_{X^*}$ which implies $\overline{B_{X^*}}^{w^*} = B_{X^*}$. Thus, $\overline{S_{X^*}}^{w^*} \subseteq \overline{B_{X^*}}^{w^*} = B_{X^*}$ so $B_{X^*} = \overline{S_{X^*}}^{w^*}$. □

**Problem 5.** *Prove that if $a_1, a_2, \ldots, a_N$ are complex numbers, then*

*(a)* $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi ikt)|^p dt \leq \sum_{k=1}^N |a_k|^p$, *if* $1 \leq p \leq 2$, *and*

*(b)* $\int_0^1 |\sum_{k=1}^N a_k \exp(2\pi ikt)|^p dt \geq \sum_{k=1}^N |a_k|^p$, *if* $2 \leq p < \infty$.

*Proof.* Note first the following facts:

- $\{\exp(2\pi ikt)\}$ is orthonormal in $L^2$
- For a finite measure space and $p \leq q$, then

$$\|f\|_p \leq \mu(X)^{1/p-1/q}\|f\|_q$$

- For a discrete $X$ and $p \leq q$, $\|f\|_q \leq \|f\|_p$.

Since $\{\exp(w\pi ikt)\}$ is orthonormal, then

$$\left\|\sum_{k=1}^{N} a_k \exp(2\pi ikt)\right\|_2^2 = \sum_{k=1}^{N} |a_k|^2.$$

Then if we let $a = (a_1, \ldots, a_N)$ and $f = \sum_{k=1}^{N} a_k \exp(2\pi ikt)$, we see that for $1 \leq p \leq 2$, we have

$$\int_0^1 \left|\sum_{k=1}^{N} a_k \exp(2\pi ikt)\right|^p dt = \|f\|_p^p \leq \|f\|_2^p = \|a\|_2^p \leq \|a\|_p^p.$$

To see (b), then similarly for $2 \leq p < \infty$,

$$\int_)^1 \left|\sum_{k=1}^{N} a_k \exp(2\pi ikt)\right|^p dt = \|f\|_p^p \geq \|f\|_2^p = \|a\|_2^p \geq \|a\|_p^p.$$

□

**Problem 6.** *Prove that if $X$ is an infinite dimensional Banach space and $X^*$ is separable in the norm topology, then there is a sequence $\{x_n\}$ of norm one vectors in $X$ such that $\{x_n\}$ converges weakly to zero.*

*Proof.* Suppose $\{x_n^*\}$ is a dense, countable subset of $X^*$.

<u>Claim:</u> For every $n$, then $\cap_{k=1}^{n} \ker(x_k^*)$ is non-trivial.

Indeed, assume to the contrary that $\cap_{k=1}^{n} \ker(x_k^*) = \{0\}$. Then the map

$$F : X \to \mathbb{F}^n$$
$$x \mapsto (x_1^*(x), \ldots, x_n^*(x))$$

is linear and injective. Let $\{e_1, \ldots, e_m\}$ be a basis for $F(X)$. Choose $y_k \in F^{-1}(\{e_k\})$. For all $x \in X$, we can write $F(x) = \sum_{i=1}^{m} a_i e_i$. so $F(x - \sum a_i y_i) = \sum a_i e_i - \sum a_i e_i = 0$ so $x$ is in the span and then $X$ must be finite dimensional, contradiction! So the claim holds.

Now, choose $x_n \in S_X \cap (\cap_{k=1}^{n} \ker(x_k))$. Fix $x^* \in X^*$, $\epsilon > 0$, so $\exists N \in \mathbb{N}$ such that $\|x^* - x_N^*\| < \epsilon$. Then for all $n \geq N$, $x_n \in \ker(x_N^*)$ so

$$|x^*(x_n)| = |(x^* - x_n^*)(x_n)| \leq \|x^* - x_N^*\| < \epsilon$$

So then $x^*(x_n) \to 0$. □

**Problem 7.** *Prove or disprove each of the following statements.*

*(a) If $\{f_n\}$ is a sequence in $C[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly.*

*Proof.* YES. Recall that $f_n \in C[0,1]$ converges weakly if and only if it converges pointwise and is uniformly bounded.

Suppose $f_n \to f$ weakly, let $M := \sup_n \|f_n\| < \infty$. Then $f_n \to f$ pointwise so $f_n^2 \to f^2$ pointwise and $\sup_n \|f_n^2\| = M^2 < \infty$. So $f_n^2 \to f^2$ weakly. □

(b) *If $\{f_n\}$ is a sequence in $L^2[0,1]$ that converges weakly, then also $\{f_n^2\}$ converges weakly. (Lebesgue measure on $[0,1]$)*

*Proof.* NO. Take $f_n(x) = x^{-1/3}\chi_{[1/n,1]}(x)$, so $f_n \to f = x^{-1/3}$ in norm but $f_n^2(x) = x^{-2/3}\chi_{[1/n,1]}(x)$ but

$$\int_0^1 f_n^2(x)x^{-1/3}dx = \int_0^1 x^{-1}\chi_{[1/n,1]} = \log(n) \to \infty.$$

□

**Problem 8.** *Let $\{f_n\}$ be a sequence of continuous functions on $\mathbb{R}$ that converges pointwise to a real valued function $f$. Prove that for each $a < b$, the function $f$ is continuous at some point of $[a,b]$.*

*Hint:* *Let $E_{n,m,k} = [|f_n - f_m| \le 1/k]$.*

*Proof.* Fix some $[a,b] \subseteq [0,1]$. By Egoroff's Theorem, $f_n \to f$ uniformly outside a set of measure $\frac{b-a}{2}$. Then $f$ must be continuous outside of this set.

*Note:* Likely, the question was meant to prove Egoroff's theorem, see Folland for that proof! □

**Problem 9.** *Let $X$ and $Y$ be compact Hausdorff spaces and let $S$ be the set of all real functions on $X \times Y$ of the form $h(x,y) = f(x)g(y)$ with $f$ in $C(X)$ and $g$ in $C(Y)$.*

*Prove or disprove that the linear span of $S$ is dense in $C(X \times Y)$.*

*Proof.* We will use Stone-Weirstrass theorem here. Note that if $h_1(x,y) = f_1(x)g_1(y)$ and $h_2(x,y) = f_2(x)g_2(y)$ are two functions in $S$, then

$$(h_1h_2)(x,y) = h_1(x,y)h_2(x,y) = f_1(x)g_1(y)f_2(x)g_2(y) = (f_1f_2)(x)(g_1g_2)(y)$$

where if $f_1, f_2 \in C(X)$ then so is $f_1f_2$ (and similarly, $g_1g_2 \in C(Y)$). So then $S$ is an algebra. Thus, it follows that $\text{span}(S)$ is an algebra as well.

Next, $S$ separates points. Indeed, suppose $(x,y) \ne (x',y')$ in $X \times Y$. If $x \ne x'$ then choose some $f \in C(X)$ that separates $x$ and $x'$. Take $g \in C(Y)$ to be the constant function $g = 1$. Then letting $h(x,y) := f(x)g(y) = f(x)$, $h$ separates the two points. If $x = x'$ then $y \ne y'$ so the same trick works, setting $f = 1 \in C(X)$ and choosing $g$ to separate $y$ and $y'$, letting $h(x,y) := f(x)g(y) = g(y)$ to then separate points.

Therefore, by the Stone-Weierstass theorem, $\text{span}(S)$ is dense in $C(X \times Y)$. □

**Problem 10.** *Let $X$ be a Hilbert space and assume that $\{x_n\}$ is a sequence in $X$ that converges weakly to zero. Prove that there is a subsequence $\{y_k\}$ of $\{x_n\}$ such that the sequence $\|N^{-1}\sum_{k=1}^N y_k\|$ converges to zero.*

*Caution: the same statement is NOT true in all Banach spaces, not even in all reflexive Banach spaces.*

*Proof. Note:* This is the Banach-Saks Theorem

We shall successively choose the $n_k$ in the following manner. Beginning for definiteness with $n_1 = 1$, let $n_2$ be the first index such that $|\langle f_1, f_n\rangle| \leq 1$ (this choice is possible since $\langle f_1, f_n\rangle \to 0$ as $n \to \infty$). In general, after having chosen $f_{n_1}, f_{n_2}, \ldots, f_{n_k}$, we choose $n_{k+1}$ so that

$$|\langle f_{n_1}, f_{n_{k+1}}\rangle| \leq \frac{1}{k}, \ldots, |\langle f_{n_k}, f_{n_{k+1}}\rangle| \leq \frac{1}{k}$$

Since $\{f_n\}$ converges weakly, then it is bounded and so $\|f_n\|$ forms a bounded sequence, say $\|f_n\| \leq M$ so by expanding the inner product, we get

$$\left\|\frac{f_{n_1} + f_{n_2} + \ldots + f_{n_k}}{k}\right\|^2 \leq \frac{kM^2 + 2 \times 1 + 4 \times \frac{1}{2} + \ldots + 2(k-1) \times \frac{1}{k-1}}{k^2} < \frac{M^2 + 2}{k}$$

which then implies

$$\left\|\frac{f_{n_1} + f_{n_2} + \ldots + f_{n_k}}{k}\right\|^2 \to 0.$$

□

**Problem 11.** *Let $F \subseteq C([0,1])$ be a family of continuous functions such that*

*(a) the derivative $f'(t)$ exists for all $t \in (0,1)$ and $f \in F$.*

*(b) $\sup_{f\in F} |f(0)| < \infty$ and $\sup_{f\in F} \sup_{t\in(0,1)} |f'(t)| < \infty$.*

*Prove that $F$ is precompact in the Banach space $C([0,1])$ equipped with the norm $\|f\| = \sup_{t\in[0,1]} |f(t)|$.*

*Proof.* We will use the Arzela-Ascoli Theorem.

To see $F$ is equicontinuous, fix some $\epsilon > 0$, and let $\delta = \frac{\epsilon}{M}$ where $M = \sup_{f\in F} \sup_{t\in(0,1)} |f'(t)| < \infty$. Then by the mean value theorem, for any $a < b$, there exists some $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$ so that $|f(b) - f(a)| \leq |f'(c)||b-a| \leq M|b-a| < M\delta = \epsilon$.

To see $F$ is pointwise bounded, we see that for any $b \in [0,1]$, then for some $c \in [0,b]$, we have $f(b) = f'(c)b + f(0)$, so that

$$|f(b)| \leq M + \sup_{f\in F} |f(0)|.$$

That is, $F$ is uniformly bounded!

Then by Arzela-Ascoli, $\overline{F}$ is compact. □

**Problem 12.** *Let $\{x_n\}$ be a weakly Cauchy sequence in a normed linear space $X$. Prove that*

*(a) $x_n$ is norm bounded in $X$*

*Proof.* Let $c$ denote the space of convergent sequences, and consider the map

$$\begin{aligned} T : X^* &\to c \\ x^* &\mapsto (x^*(x_n)) \end{aligned}$$

By the Uniform Boundedness Principle, $T$ is closed. Then by the closed graph theorem, $T$ is continuous, so $\|T\| < \infty$. By Hahn-Banach Theorem, $\|T\| = \sup_n \|x_n\|$. □

*(b) There exists $x^{**}$ in $X^{**}$ such that $x_n$ converges weak* to $x^{**}$, and $\|x^{**}\| \leq \liminf \|x_n\|$.*

*Proof.* Since $(x_n)$ is weakly Cauchy, then for every $x^* \in X^*$ the sequence $(x^*(x_n))$ is Cauchy, hence convergent. We can define

$$\begin{aligned} x^{**} : X^{**} &\to \mathbb{C} \\ x^* &\mapsto \lim_n x^*(x_n) \end{aligned}$$

Uniform boundedness shows that $\|x_n\|$ is bounded, hence $x^{**}$ is bounded. Finally,

$$|x^{**}(x^*)| = \liminf |x^*(x_n)| \leq \liminf \|x^*\|\|x_n\| = \Big(\liminf \|x_n\|\Big)\Big(\|x^*\|\Big).$$

So then $\|x^{**}\| \leq \liminf \|x_n\|$. □

# 6 January 2017

**Problem 1.** *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Prove directly from the definition of convergence almost everywhere that if for all $n$, $\mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) < n^{-3/2}$, then $f_n \to 0$ $\mu$-a.e.*

*Proof.* Let $E = \{x \in \Omega \mid \lim_n |f_n(x)| = 0\}$. We want $\mu(E^c) = 0$. Let

$$M = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > 1/n\}$$

Since

$$\mu\left(\bigcup_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\}\right) \leq \sum_{n=m}^{\infty} \mu\left(\left\{x \in \Omega \mid |f_n(x)| > \frac{1}{n}\right\}\right) < \sum_{n=m}^{\infty} n^{-3/2} \to 0.$$

Therefore, $\mu(M) = 0$ and

$$M^c = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \{x \in \Omega \mid |f_n(x)| \leq 1/n\}.$$

Note: $f_n(x) \to 0$ if and only if $\forall \epsilon > 0$, $\exists N$ s.t. $\forall n > N$, $|f_n(x)| < \epsilon$.

So for any $x \in M^c$ choose $1/N < \epsilon$ s.t. $\forall n > N$ we have $|f_n(x)| \leq \frac{1}{n} < \frac{1}{N} < \epsilon$.

Therefore $M^c \subseteq E$, so $E^c \subseteq M$, implying $\mu(E^c) = 0$. □

**Problem 2.** *Find all $f$ in $L^1(1,2)$ such that for every natural number $n$ we have $\int_1^2 x^{2n} f(x)dx = 0$. Give reasons for all assertions you make.*

*Proof.* Let $f(x) = 0$ on $x = 1, 2$. We now consider $f \in L^1[1,2]$. Using Stone-Weierstrass to show we can pass to the case $\int_1^2 g(x)f(x) = 0$ for all $g \in \mathcal{C}[1,2]$.

By a standard density argument, we may pass to the case where $g$ is a step function. We claim that $f = 0$ a.e.

Assume not. WLOG there exists some $E = \{x \in [1,2] \mid f(x) > 0\}$ with $m(E) > 0$ (else consider $-f$).

Since $f \in L^1[1,2]$ then $E_\infty := \{x \in [1,2] \mid f(x) = \infty\}$ is a null set. Define $E_n := \{x \in [1,2] \mid 1/n < f(x) < n\}$. We can write $E = (\bigcup_n E_n) \cup E_\infty$. So there exists some $N$ such that $m(E_N) = a > 0$.

We can write $A$ as a finite disjoint union of open intervals, $A = \sqcup_{i=1}^m I_i$, such that $m(E_N \triangle A) < \epsilon$ and $A \subseteq E_N$.

Put $g = \sum_{i=1}^m \chi_{I_i}$, then $\int_1^2 g(x)f(x) = \int_{E_N} f(x)dx$. Since

$$\left| \int_{E_N} f(x) - \int_A f(x) \right| \leq N m(E_N \triangle A) < N\epsilon$$

If we choose $\epsilon$ small enough, we see the contradiction since $\int_1^2 g(x)f(x) > 0$. □

**Problem 3.** *A. Prove that there exists a sequence of measurable functions $g_n$ on $[0,1]$ such that*

*(a) $g_n(x) \geq 0$ for any $x \in [0,1]$;*

*(b) $\lim_n g_n(x) = 0$ a.e.;*

*(c) For any continuous function $f \in C[0,1]$,*

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx = \int_0^1 f(x)dx.$$

*Proof. (Solution from Ting Lu, TeX-ed by John Weeks)*

It suffices to assume $f$ is non-negative. Any $f \in C[0,1]$ is uniformly continuous since $[0,1]$ is compact. The following lemma will then come in handy:

**Claim.** With $f$ as above, $n \int_{[x, x+\frac{1}{n^2}]} (f(y) - f(x))dy = o(\frac{1}{n})$.

For any $\epsilon > 0$, $\exists N \in \mathcal{N}$ such that $0 < x < \frac{1}{N}$ then $f(x) - f(0) < \varepsilon$. Hence for $n > N$, $n\int_{[0,\frac{1}{n^2}]}(f(x) - f(0)) < n\epsilon\frac{1}{n^2}$.

A clear extension to this is that $\int_{[x,x+\frac{1}{n}]}(f(y) - f(x))dy = o(\frac{1}{n})$.

Let $g_n(x) := \sum_{k=0}^{n-1} n\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]}$, and let $\varepsilon, N$ be as above. Clearly $g_n$ is measurable and satisfies (a) and (b) above. To prove (c), observe that, for $n > N$,

$$\begin{aligned} I_k &:= \left| \int_{\frac{k}{n}}^{\frac{k+1}{n}} nf\chi_{[\frac{k}{n},\frac{k}{n}+\frac{1}{n^2}]} - f \right| \\ &= \left| n\int_{\frac{k}{n}}^{\frac{k}{n}+\frac{1}{n^2}} f - \int_{\frac{k}{n}}^{\frac{k+1}{n}} f \right| \\ &= o\left(\frac{1}{n}\right). \end{aligned}$$

Hence $|\int(fg_n - f)| \leq \sum_{k=0}^{n-1} I_k = o(1)$. □

B. *If $g_n$ is a sequence of measurable functions on $[0,1]$ such that (a), (b), and (c) are satisfied, what can you say about $\int_0^1 \sup_n g_n(x)dx$?*

*Proof.* here □

**Problem 4.** *We say that a sequence $\{a_n\}_{n=1}^{\infty}$ in $[0,1]$ is equidistributed (in $[0,1]$) if and only if for all intervals $[c,d] \subset [0,1]$,*

$$\lim_{n\to\infty} \frac{|\{a_1,\dots,a_n\} \cap [c,d]|}{n} = d - c.$$

*(Here $|A|$ denotes the number of elements in the set $A$.)*

*Let $\mu_N = \frac{1}{N}\sum_{1\leq n\leq N}\delta_{a_n}$ with $\delta_{a_n}$ the point measure at $a_n$, that is, for any subset $S \in [0,1]$, $\delta_{a_n}(S) = \begin{cases} 1 & \text{if } a_n \in S \\ 0 & \text{if } a_n \notin S \end{cases}$.*

*Show that $\{a_n\} \subset [0,1]$ is equidistributed if and only if*

$$\lim_{N\to\infty}\int_0^1 fd\mu_N = \int_0^1 fdm,$$

*for all continuous functions on $[0,1]$, where $m$ is Lebesgue measure.*

*Proof.* Note that $\{a_n\}$ is equidistributed if and only if

$$\lim_{n} \frac{|\{a_1,\dots,a_n\} \cap [c,d]|}{n} = d - c$$

if and only if

$$\lim_n \int_0^1 f d\mu_N = \int_0^1 f dm \quad \text{for } f \text{ simple functions (since we can take } f = \chi_{[c,d]})$$

⇒) It's easy to see if $\{a_n\}$ is equidistributed for $f = \chi_{[c,d]}$.

$$\lim_N \int_0^1 f d\mu_N = \lim_N \frac{|\{a_1, \ldots, a_N\} \cap [c,d]|}{N} = d - c = \int_0^1 f dm$$

Thus, "=" holds for step functions.

Using Darboux's definition of integral for $f \in C[0,1]$, $\forall \epsilon > 0$ there exists step functions $f_1, f_2$ such that $f_1 \leq f \leq f_2$ and $\int_0^1 (f_2 - f_1)dx < \epsilon$ where the lower sum is

$$\int_0^1 f_1(x)dx = \lim_N \frac{1}{N} \sum_1^N f_1(a_n) \leq \liminf_N \frac{1}{N} \sum_1^N f(a_n)$$

and the upper sum is

$$\int_0^1 f_2(x)dx = \lim_N \frac{1}{N} \sum_1^N f_2(a_n) \geq \limsup_N \frac{1}{N} \sum_1^N f(a_n)$$

Then

$$\left| \limsup_N \frac{1}{N} \sum_1^N f(a_n) - \liminf_N \frac{1}{N} \sum_1^N f(a_n) \right| \leq \epsilon.$$

Therefore $\lim_N \frac{1}{N} \sum_1^N f(a_n)$ exists and by definition must be $\int_0^1 f d\mu$.

⇐) If we know $\lim_K \int_0^1 g_n d\mu_K = \int_0^1 g_n d\mu$ for all $g_n \in C[0,1]$. Let $f = \chi_{[c,d]}$, choose $g_n \to f$ in $L^1$ and each $g_n \searrow f$ positive, $g_n \in C[0,1]$ with $g_n|_{[c,d]} = 1 = f|_{[c,d]}$.

We want to show $\lim_K \int_0^1 f d\mu_K = \int_0^1 f dm$. Indeed,

$$\begin{aligned}
\left|\int_0^1 f d\mu_K - \int_0^1 f dm\right| &= \left|\int_c^d f d\mu_K - \int_c^d f dm\right| \\
&= \left|\int_c^d g_n d\mu_K - \int_c^d f dm\right| \\
&= \int_c^d g_n d\mu_K - \int_c^d f dm \\
&\leq \int_0^1 g_n d\mu_K - \int_0^1 f dm \\
&\leq \left|\int_0^1 g_n d\mu_K - \int_0^1 g_n dm\right| + \left|\int_0^1 g_n dm - \int_0^1 f dm\right| \to 0.
\end{aligned}$$

□

**Problem 5.** *Consider the space $C([0,1])$ of real-valued continuous functions on the unit interval $[0,1]$. We denote by $\|f\|_\infty := \sup_{x\in[0,1]} |f(x)|$ the supremum norm of $f \in C([0,1])$ and by $\|f\|_2 := (\int_0^1 |f(x)|^2 dx)^{\frac{1}{2}}$ the $L^2$-norm of $f \in C([0,1])$. Let $S$ be a closed linear subspace of $(C([0,1]), \|\cdot\|_\infty)$. Show that if $S$ is complete in the norm $\|\cdot\|_2$, then $S$ is finite-dimensional.*

*Proof.* Let $T : (S, \|\cdot\|_2) \to (S, \|\cdot\|_\infty)$ by $T(x) = x$. Note that both spaces are complete.

Assume $x_n \to x$ in $\|\cdot\|_2$ and $T(x_n) \to y$ in $\|\cdot\|_\infty$ then

$$\|T(x_n) - y\|_2 \leq \|T(x_n) - y\|_\infty \to 0.$$

So

$$\|x - y\|_2 \leq \|x - T(x_n)\|_2 + \|T(x_n) - y\|_2 \leq \|x - x_n\|_2 + \|T(x_n) - y\|_\infty \to 0$$

so $x = T(x) = y$.

Therefore, by closed graph theorem, we know $T$ is bounded. So there exists some $C$ such that $\|f\|_\infty \leq C\|f\|_2$.

Now let $f_1, \dots, f_n$ be an orthonormal family in $S$. Then for all fixed $x \in [0,1]$

$$f_1(x)^2 + \cdots + f_n(x)^2 \leq \|f_1(x)f_1 + \cdots + f_n(x)f_n\|_\infty \leq C\|f_1(x)f_1 + \cdots + f_n(x)f_n\|_2$$

So then because $f_n$'s are orthogonal and $\|f_k\|_2^2 = 1$,

$$(f_1(x)^2 + \cdots + f_n(x)^2)^2 \leq C^2 \left(f_1(x)^2\|f_1\|_2^2 + \cdots + f_n(x)^2\|f_n\|_2^2\right) = C^2(f_1(x)^2 + \cdots + f_n(x)^2)$$

Then $f_1(x)^2 + \cdots + f_n(x)^2 \le C^2$. So

$$n = \int_0^1 f_1(x)^2 + \cdots + f_n(x)^2 dx \le \int_0^1 C^2 dx = C^2 \Rightarrow n \le C^2$$

Thus, the number of orthogonal family in $S$ is at most $C^2$. So $S$ is finite dimensional. □

**Problem 6.** *Prove that if a function $f : [0,1] \to \mathbb{R}$ is Lipschitz, with*

$$|f(x) - f(y)| \le M|x - y|$$

*for all $x, y \in [0,1]$, then there is a sequence of continuously differentialbe functions $f_n : [0,1] \to \mathbb{R}$ such that*

*(i)* $|f_n'(x)| \le M$ *for all* $x \in [0,1]$*;*

*(ii)* $f_n(x) \to f(x)$ *for all* $x \in [0,1]$.

*Proof.* It's easy to prove $f$ is aboslutely continuous $\Rightarrow$ $f$ is of bounded variable $\Rightarrow$ $f$ is differentiable a.e. $\Rightarrow$ $f'$ exists a.e.

Also, when $f'$ exists, $|f'(x)| \le M$.

Then there exists simple $\phi_1, \phi_2, \dots$ such that $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |\phi_n| \le \cdots \le |f'| \le M$ and $\phi_n \to f'$ uniformly on $[0,1]$ where $f'$ exists. Define

$$f_n(x) := \int_0^x \phi_n(t)dt + f(0) \qquad f(x) := \int_0^x f'(t)dt + f(0)$$

Then $|f_n'(x)| = |\phi_n(x)| \le M$ and for all $x \in [0,1]$,

$$|f_n(x) - f(x)| \le \int_0^x |\phi_n(t) - f'(t)|dt \to 0$$

since $\phi_n$ converges to $f'$ uniformly. □

**Problem 7.** *Given $f : \mathbb{R} \to \mathbb{R}$ bounded and uniformly continuous and $K_n$ with $K_n \in L^1(\mathbb{R})$ for $n = 1, 2, 3, \dots$ such that*

*(i)* $\|K_n\|_1 \le M < \infty$, $n = 1, 2, 3, \dots$

*(ii)* $\int_{-\infty}^{\infty} K_n(x)dx \to 1$ *as* $n \to \infty$.

*(iii)* $\int_{\{x||x|>\delta\}} |K_n(x)| \to 0$ *as* $n \to \infty$ *for all* $\delta > 0$.

*Show that $K_n * f \to f$ uniformly, where*

$$K_n * f(x) = \int_{-\infty}^{\infty} K_n(y) f(x-y) dy.$$

*Proof.* For all $x \in \mathbb{R}$,

$$\begin{aligned}
|K_n * f(x) - f(x)| &\leq \left| K_n * f(x) - \int_{-\infty}^{\infty} K_n(y) f(x) dy \right| + \left| \int_{-\infty}^{\infty} K_n(y) f(x) dy - f(x) \right| \\
&\leq \int_{-\infty}^{\infty} |K_n(y)||f(x-y) - f(x)| dy + \|f\|_\infty \left| \int_{-\infty}^{\infty} K_n(y) dy - 1 \right| \\
&\leq \int_{-\infty}^{\infty} |K_n(y)||f(x-y) - f(x)| dy + c\epsilon \\
&= \int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)| dy + \int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)| dy + c\epsilon
\end{aligned}$$

For $\int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)| dy$, by uniform continuity we ahve

$$\int_{B(0,\delta)} |K_n(y)||f(x-y) - f(x)| dy \leq \epsilon \int_{B(0,\delta)} |K_n(y)| dy \leq \epsilon \|K_n\|_1 < \epsilon M$$

For $\int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)| dy$, by the third assumption we have

$$\int_{B(0,\delta)^c} |K_n(y)||f(x-y) - f(x)| dy \leq 2\|f\|_\infty \int_{B(0,\delta)^c} |K_n(y)| dy \leq 2C\epsilon$$

Let $\epsilon \to 0$, so we've got it. □

**Problem 8.** *(a) Construct a Lebesgue measurable subset $A$ of $\mathbb{R}$ so that for all reals $a < b$, $0 < m(A \cap [a, b]) < b - a$ where $m$ is Lebesgue measure on $\mathbb{R}$.*

*Proof.* Enumerate all rational intervals $I_1, I_2, \ldots$. For each $I_n$, construct a fat Cantor set $N_n \subseteq I_n$ with positive measure.

Since $N_n$ is nowhere dense, there exists some interval $\widetilde{I_n} \subseteq I_n$ and $\widetilde{I_n} \cap N_n = \emptyset$.

Construct another fat Cantor set $M_n \subseteq \widetilde{I_n}$ and define $A := \cup M_n$.

Now, for all $I = [a, b]$ there exists some $n$ such that $N_n \subseteq I_n \subseteq I$ with $N_n \cap A = \emptyset$ (can be done by induction). We see $m(A \cap I) \geq m(M_n) > 0$ and

$$m(A \cap I) < m(I \backslash N_n) = m(I) - m(N_n) < m(I) = b - a.$$

□

*(b) Suppose $A \subseteq \mathbb{R}$ is a Lebesgue measurable set and assume that*

$$m(A \cap (a, b)) \leq \frac{b-a}{2}$$

*for any $a, b \in \mathbb{R}$, $a < b$. Prove that $\mu(A) = 0$.*

*Proof.* Consider an open set $U \supseteq A$ with $m(U \backslash A) < \epsilon$. Then $U = \sqcup_{i=1}^{\infty}(a_i, b_i)$ and measurable. So

$$m(U) = m(A \cap U) + m(U \cap A^c) < m(A \cap U) + \epsilon$$

Since

$$m(A \cap U) = m\left(A \cap (\sqcup_{i=1}^{\infty}(a_i, b_i))\right) = \sum_{i=1}^{\infty} m(A \cap (a_i, b_i)) \leq \sum_{i=1}^{\infty} \frac{b_i - a_i}{2} = \frac{1}{2} m(U)$$

then

$$m(U) < \frac{1}{2} m(U) + \epsilon \quad \Rightarrow \quad m(U) < 2\epsilon \quad \Rightarrow \quad m(A) \leq m(U) \to 0$$

□

**Problem 9.** *Prove or disprove that the unit ball of $L^7(0,1)$ is norm closed in $L^1(0,1)$.*

*Proof.* Let

$$B := \left\{ f \mid \int_0^1 |f|^7 dx \leq 1 \right\}.$$

Let $\{f_n\} \subseteq B$ such that $f_n \to f$ in $L^1$. We want to show $f \in B \Leftrightarrow \int_0^1 |f|^7 dx \leq 1$.

Since $f_n \to f$ in $L^1$, then $f_n \to f$ in measure. Thus, there exists a subsequence $f_{n_k}$ such that $f_{n_k} \to f$ a.e.

Therefore, $|f_{n_k}|^7 \to |f|^7$ a.e.. By Fatou's Lemma,

$$\int_0^1 |f|^7 dx \leq \liminf_k \int_0^1 |f_{n_k}|^7 dx \leq 1.$$

□

**Problem 10.** *Let $C$ be the Banach space of convergent sequences of real numbers under the supremum norm. Compute the extreme points of the closed unit ball, $B$, of $C$ and determine whether $B$ is the closed convex hull of its extreme points.*

*Proof.* If $|x(m)| < 1$ for some $m$ then there exists $\delta > 0$ such that $|x(m) - \delta| \leq 1$, $|x(m) + \delta| \leq 1$.

Define $y_1, y_2 \in B$ such that

$$y_1(n) = x(n) \text{ for } n \neq m \text{ and } y_1(m) = x(m) + \delta$$
$$y_2(n) = x(n) \text{ for } n \neq m \text{ and } y_2(m) = x(m) - \delta$$

Then $y_1 \neq y_2$ and $x = \frac{1}{2}(y_1 + y_2)$ so $x$ is not an extreme point.

If $|x(n)| = 1$ for all $n$, if $x = \lambda y_1 + (1-\lambda)y_2$ for $y_1 \neq y_2 \in B$ then since $|y_i(n)| \leq n$,

$$|x(n)| = 1 = |\lambda y_1(n) + (1-\lambda)y_2(n)| \leq \lambda|y_1(n)| + (1-\lambda)|y_2(n)| \leq 1$$

Equality holds only when $y_1(n) = y_2(n) = \pm 1$. So $y_1 = y_2$ so $x$ is indeed an extreme point. Also, $x$ needs to be convergent so

$$\text{Ext}(B) = \{x \mid |x(n)| = 1 \ \exists N \text{ s.t. } x(n) = 1 \text{ or } -1 \text{ for all } n > N\}.$$

Problem in my proof of determining whether $B$ is the closed convex hull of it's extreme points. □

**Problem 11.** *Show that every convex continuous function defined on the convex unit ball of a reflexive Banach space achieves a minimum. (A convex function on a convex subset $A$ of a normed space is a real valued function, $f$, on $A$ s.t. for every $x, y \in A$ and every $0 < \lambda < 1$ we have $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$.)*

*Proof.* Recall the following classical result in convex analysis: $f$ is lower-semicontinuous convex $\Leftrightarrow$ $f$ is weakly lower-semicontinuous convex $\Rightarrow$ $f$ can achieve minimum (since the unit ball is weak-compact).

Then by Alaoglu, closed unit ball of a reflexive Banach space is weak-compact = weak*-compact.

here □

# 7 August 2016

**Problem 1.** *Let $\mathcal{A}$ be the set of all real valued functions on $[0,1]$ for which $f(0) = 0$ and $|f(t) - f(s)|^{1/2} \leq t - s$ for all $0 \leq s < t \leq 1$*

*(a) Prove that $\mathcal{A}$ is a compact subset of $C[0,1]$.*

*Proof.* It should be clear to the reader that this question requires Arezela-Ascoli Theorem. To see $\mathcal{A}$ is equicontinuous, fix $x \in [0,1]$ and $\epsilon > 0$. Then for $y \in B(\sqrt{\epsilon}, x)$,

$$|f(x) - f(y)| \leq |x-y|^2 < \epsilon$$

For pointwise bounded, for $x \in [0,1]$ then $|f(x)|^{1/2} = |f(x) - f(0)|^{1/2} \leq x$ implies $|f(x)| \leq x^2$.

To see $\mathcal{A}$ is closed, take a sequence $\{f_n\} \subseteq \mathcal{A}$ such that $f_n \to f$ (i.e. for all open $U$ containing $f$, there exists $N$ such that for all $n \geq N$, $f_n \in U$), then

$$|f(t) - f(s)| \leq |f(t) - f_n(t)| + |f_n(t) - f_n(s)| + |f_n(s) - f(s)| < 2\epsilon + |t-s|^2$$

This holds for all $\epsilon > 0$ so $|f(t) - f(s)| \leq |t-s|^2$.

Clearly, $f(0) = 0$ so $f \in \mathcal{A}$. Thus $\mathcal{A}$ is closed so by Arzela-Ascoli, $\mathcal{A}$ is compact in $\mathcal{C}[0,1]$. □

*(b) Prove that $\mathcal{A}$ is a compact subset of $L_1[0,1]$*

*Proof.* Consider the map

$$\begin{aligned} \text{id} : C[0,1] &\to L^1[0,1] \\ f &\mapsto f \end{aligned}$$

Since $\| \,\text{id}\, \|_1 = \int_0^1 |f| dx \leq \|f\|_\infty$, id is a bounded map.

From (a), $\mathcal{A}$ is compact in $C[0,1]$ so $\text{id}(\mathcal{A}) = \mathcal{A} \subseteq L^1[0,1]$ is also compact.

<u>Remark:</u> $\mathcal{A}$ is also closed in $L^1[0,1]$ since all compact subsets of a metric space is closed. □

**Problem 2.** *(a) Let $f(x)$ be a real valued function on the real line that is differentiable almost everywhere. Prove that $f'(x)$ is a Lebesgue measurable function.*

*Proof.* Let

$$f_n 9x) = \frac{f(x+1/n) - f(x)}{1/n}$$

so $f_n \to f'$ almost everywhere. Since $f$ is differentiable almost everywhere, then $f$ is continuous almost everywhere.

<u>Claim:</u> $f$ is Lebesgue measurable

Let $D = \{\text{all discontinuities of } f\}$ so $m(D) = 0$ and $D$ is measurable. Let $E = D^c = \{x \mid f \text{ is continuous at } x\}$ so $E$ is measurable too.

$$f^{-1}((a,\infty)) = f^{-1}((a,\infty) \cap E) \cup f^{-1}((a,\infty) \cap E^c)$$

Since $f|_E$ is continuous, $f^{-1}(a,\infty) \cap E = f|_E^{-1}(a,\infty)$ is open in $E$. So $f^{-1}(a,\infty) \cap E = U \cap E$ for some open set $U \subseteq \mathbb{R}$. Then $f^{-1}(a,\infty) \cap E$ is measurable.

Now $f^{-1}(a,\infty) \cap E^c \subseteq E^c$, so completeness implies $f^{-1}(a,\infty) \cap E^c$ is measurable. Thus, $f$ is Lebesgue measurable so the claim holds.

So each $f_n$ is measurable, thus $f' = \lim f_n$ almost everywhere is also Lebesgue measurable. □

*(b) Prove that if $f$ is a continuous real valued function on the real line, then the set of points at which $f$ is differentiable is measurable.*

*Proof.* Let

$$F(x,h) = \frac{f(x+h) - f(x)}{h}$$

which is continuous on $\mathbb{R} \times (\mathbb{R} \backslash \{0\})$. If $x$ is a differentiable point of $f$, then for all $\epsilon > 0$, there exists a $\delta > 0$ and some $Y$ such that for all $h$ with $|h| < \delta$, we have $|F(x,h) - Y| < \epsilon$. i.e.

$$D = \{x \mid \text{differentiable point of } f\} = \bigcap_{\epsilon} \bigcup_{\delta} \bigcup_{Y} \bigcap_{|h|<\epsilon} \{x \mid |F(x,h) - Y| < \epsilon\}$$

For fixed $\epsilon, \delta, Y, h$ then $\{x \mid |F(x,h) - Y| < \epsilon\}$ is open, thus Borel.

By taking only rational $\epsilon, \delta, Y, h$ we have $D$ Borel measurable. □

**Problem 3.** *(a) Let $f$ be a real valued function on the unit interval $[0,1]$. Prove that the set of points at which $f$ is discontinuous is a countable union of closed subsets.*

*Proof.* $f$ is continuous at $p$ if for all $n$, there exists an open $U$ containing $p$ such that $|f(x) - f(y)| < 1/n$ for all $x, y \in U$. Fix $n$ and let

$$V_n = \bigcup_p \{p \text{ s.t. there exists an appropriate } U\} = \bigcup \{\text{appropriate } U\}$$

Hence, $V_n$ is open. Then

$$\{\text{points where } f \text{ is continuous}\} = \bigcap_n V_n$$

So $\{\text{points where } f \text{ is discontinuous}\} = \bigcup_n V_n^c$ where $V_n^c$ is closed. □

*(b) Prove that there does not exist a real valued function on $[0,1]$ that is continuous at all rational points but discontinuous at all irrational points.*

*Proof.* By (a), the irrational poitns would be a countable union of closed subsets. Note that because any open set in $[0,1]$ contains a rational point, then if $\mathbb{Q}^c_{[0,1]} = \bigcup_n F_n$ where $F_n$ is closed and $F_n^\circ = \emptyset$. Then

$$[0,1] = \mathbb{Q}_{[0,1]} \cup \mathbb{Q}^c_{[0,1]} = \left( \bigcup_{q \in \mathbb{Q}} \{q\} \right) \cup \left( \bigcup_n F_n \right)$$

So $[0,1]$ is a countable union of nowhere dense sets. This contradicts Baire-Category Theorem. □

**Problem 4.** *Let $(\Omega, \mathcal{A}, \mu)$ be a finite measure space and let $(f_n)$ be a sequence of measurable functions on $X$ that converges pointwise to zero. Prove that $(f_n)$ converges in measure to zero. Show that the converse is false for $[0,1]$ with Lebesgue measure.*

*Proof.* Fix $\epsilon > 0$. To show $\mu(\{x \mid |f_n(x)| > \epsilon\}) \to 0$, we need $\forall m\ \exists N_m$ such that $\forall n \geq N_m$, $\mu(\{x \mid f_n(x)| > \epsilon\}) < 1/m$.

By Egoroff's Theorem, there exists some $E \subseteq X$ with $\mu(E) < 1/m$ and $f_n \rightrightarrows 0$ uniformly on $E^c$.

Thus, $\exists N_m$ such that for $n \geq N_m$ $|f_n(x)| < \epsilon$ for all $x \in E^c$ so

$$\mu(\{x \mid |f_n(x)| > \epsilon\}) \leq \mu(E) < \frac{1}{m} \qquad \forall n \geq N_m$$

Thus, $\mu(\{x \mid |f_n(x)| > \epsilon\}) \to 0$.

**Counterexample:** Let $f_1 = \chi_{[0,1]}, f_2 = \chi_{[0,1/2]}, f_3 = \chi_{[1/2,1]}, \dots, f_n = \chi_{[j/2^k,(j+1)/2^k]}$ for $n = 2^k + j, 0 \le j < 2^k$.

So $f_n$ does not approach 0 pointwise, but $f_n \to 0$ in $L^1$, hence in measure. ☐

**Problem 5.** *If $f$ is Lebesgue integrable on the real line, prove that* $\lim_{h\to 0} \int_{\mathbb{R}} |f(x+h)-f(x)|dx = 0$.

*Proof.* Recall: the set $C_c(\mathbb{R})$ of continuous, compactly supported functions is dense in $L^1(\mathbb{R})$.

Fix $\epsilon > 0$ and find $g \in C_c(\mathbb{R})$ with $\|f-g\|_1 < \epsilon$. Since $g$ is continuous, $\lim_n |g(x+1/n) - g(x)| = 0$ ofr all $x$.

Since $g$ is compactly supported, then there exists some compact $K$ such that $\text{supp}(g) \subseteq K$.

So there exists a compact $K'$ such that $\text{supp}(g) \cup \text{supp}(g(x+1/n)) \subseteq K'$ for all $n$ (this follows from $1/n \ge 1$ for all $n$ since we can take $K' = \{k + x \mid k \in K, x \in [0,1]\}$).

Dini's theorem implies that $|g(x+1/n) - g(x)| \rightrightarrows 0$ so

$$\int \mathbb{R}|g(x+1/n) - g(x)|dx = \int_{K'} |g(x+1/n) - g(x)|dx \to 0$$

So then

$$\begin{aligned} &\int_{\mathbb{R}} |f(x+1/n) - f(x)|dx \\ \le &\int_{\mathbb{R}} |f(x+1/n) - g(x+1/n)|dx + \int_{\mathbb{R}} |g(x+1/n) - g(x)|dx + \int_{\mathbb{R}} |g(x) - f(x)|dx \\ &< 2\epsilon + \int_{\mathbb{R}} |g(x+1/n) - g(x)|dx \to 2\epsilon \end{aligned}$$

Since it holds for all $\epsilon > 0$ then $\lim_n \int_{\mathbb{R}} |f(x+1/n) - f(x)|dx = 0$. ☐

**Problem 6.** *Prove or disprove that there exists a sequence $(P_n)$ of polynomials such that $(P_n(t))$ converges to one for every $t \in [0,1]$ but $\int_0^1 P_n(t)dt$ converges to two as $n \to \infty$.*

*Proof.* Consider

$$f_n(x) = \begin{cases} n^2 x & x \in [0, 1/n] \\ -n^2 x + 2n + 1 & x \in [1/n, 2/n] \\ 1 & x \in [2/n, 1] \end{cases}$$

(that is, $f_n$ linearly connects the points $(0,1), (1/n, n+1), (2/n, 1), (1,1)$.)

So $f_n(x) \to 0$ for all $x \in [0,1]$ but $\int_0^1 f_n(x)dx = 2$.

Then by Stone-Weierstrass, we can find polynomials $P_n$ such that $\|f_n - P_n\|_\infty \leq 2^{-n}$. Then $\forall x$

$$|P_n(x) - 1| \leq |P_n(x) - f_n(x)| + |f_n(x) - 1| \to 0$$

and $\int_0^1 |f_n(x) - P_n(x)|dx \to 0$ so $\int_0^1 P_n(x)dx \to 2$. □

**Problem 7.** *Let $(f_n)$ be a uniformly bounded sequence of continuous functions on $[0,1]$ that converges pointwise to zero. Prove that 0 is in the norm closure in $C[0,1]$ of the convex hull of $(f_n)$ (the norm is of course the sup norm on $C[0,1]$).*

*Proof.* By the Geometrical version of the Hahn-Banach,

$$\overline{\text{conv}\{f_n\}}^{\text{weak}} = \overline{\text{conv}\{f_n\}}^{\|\cdot\|}$$

We just need to show that $0 \in \overline{\text{conv}\{f_n\}}^{\text{weak}}$. By Riesz-Representation Theorem, $C[0,1]^* = \mathcal{M}[0,1]$. For all $\mu \in M[0,1]$,

$$\left| \int_{[0,1]} f_n d\mu \right| \leq \int_{[0,1]} |f_n| d|\mu| \to 0$$

by Dominated Convergence Theorem. Thus, $f_n \to 0$ weakly. □

**Problem 8.** *Assume that $X$ is a reflexive Banach space and $\phi$ is a continuous linear functional on $X$. Prove that $\phi$ achieves its norm; that is, prove that there is a norm one vector $x$ in $X$ such that $\phi(x) = \|x\|$. Show by example that there is a continuous linear functional on the Banach space $\ell_1$ that does not achieve its norm.*

*Proof.* Recall: $X$ reflexive $\Rightarrow \overline{B_X}$ is weak-compact $\Rightarrow \overline{B_X}$ is weak-sequentially compact.

There exists a sequence $\{x_n\} \subseteq \overline{B_X}$ such that $\phi(x_n) \nearrow \|\phi\|$.

Choose a weakly-convergent subsequence $\{x_{n_k}\}$ that converges to $x \in \overline{B_X}$. Then for all $\varphi \in X^*$, $\varphi(x_{n_k}) \to \varphi(x)$.

In particular,

$$\|\phi\| = \lim_n \phi(x_n) = \lim_k \phi(x_{n_k}) = \phi(x).$$

*Alternative Proof.* For all $\phi \in X^*$, by Hahn-Banach Separation Theorem, there exists some $x^{**} \in X^{**}$ such that $\|x^{**}\|_{X^{**}} = 1$ and $x^{**}(\phi) = \|\phi\|_{X^*}$.

Since $X$ is reflexive, $\exists x \in X$ such that $\hat{x} = x^{**}$ so

$$\|\phi\|_{X^*} = x^{**}(\phi) = \hat{x}(\phi) = \phi(x).$$

**Counterexample:** Choose $y = (1 - 1/n)_n \in \ell^\infty$. Then $\forall x \in \ell_1$,

$$y(x) = \left|\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) x(n)\right| \leq \sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right) |x(n)| < \sum_{n=1}^{\infty} |x(n)| = \|x\|_1 = 1 = \|y\|_\infty.$$

□

**Problem 9.** *Suppose that $X$ is a non separable Banach space. Prove that there is an uncountable subset $A$ of the unit ball of $X$ such that for all $x \neq u$ in $X$, $\|x - y\| > 0.9$.*

*Proof.* By transfinite induction, construct $(x_\alpha)\alpha < \omega_1 \subseteq \overline{B_X}$ where $\omega_1$ is the uncountable ordinal.

Given $\alpha < \omega_1$, let $U_\alpha := \overline{\text{span}\{x_\beta \mid \beta < \alpha\}}$ which is separable.

Since $X$ is not separable, $U_\alpha \subsetneq X$.

By Riesz-Lemma, there exists $\|x_\alpha\| = 1$ such that $d(x_\alpha, U_\alpha) \geq 1 - \epsilon$ (put $\epsilon > 0.1$).

So $(x_\alpha)$ satisfies $\|x_\alpha - x_\beta\| \geq 0.9$ and is uncountable.

*Alternative Proof if it were not restricted to $B_X$.* Fix $r > 0$. Zornicate over all subsets $A \subseteq X$ such that $\forall x \neq y$, $\|x - y\| > r$.

Find a maximal subset $A_r \subseteq X$ as above. If $A_r$ is uncountable, by scaling of $r$, we're done.

Suppose not, so each $A_r$ is countable. Enumerate as $\{x_n^r\}_n$. By maximality, for all $x \in X$, $\forall \epsilon > 0$ if $r > 1/\epsilon$ then there exists $n \in \mathbb{N}$ such taht $\|x - x_n^m\| < r < \epsilon$ (i.e. $d(x, A_r) < r, \forall x \in X$).

Let $A = \bigcup_{q \in \mathbb{Q}} A_q$ so $A$ is a countable dense subset of $X$. Contradiction!

Therefore, there exists $q \in \mathbb{Q}$ such that $A_q$ is uncountable. Consider $A' = \{\frac{0.9}{q} x \mid x \in A_q\}$ so for all $x', y' \in A$,

$$\|x' - y'\| = \left\|\frac{0.9}{q} x - \frac{0.9}{q} y\right\| = \frac{0.9}{q} \|x - y\| > 0.9$$

Thus, there eixsts an uncountable $A \subseteq X$ such that for all $x, y \in A$, $\|x - y\| > 0.9$. □

**Problem 10.** *If $A$ is a Borel subset of the line, then $E = \{(x, y) \mid x - y \in A\}$ is a Borel subset of the plane. If the Lebesgue measure of $A$ is 0, then the Lebesgue measure of $E$ is 0.*

*Proof.* Define $f(x, y) = x - y : \mathbb{R}^2 \to \mathbb{R}$. This is continuous. Let

$$\mathcal{A} := \{S \subseteq \mathbb{R} \mid f^{-1}(S) \text{ is a Borel set of } \mathbb{R}^2\}$$

Then $\mathcal{A}$ is a $\sigma$-algebra (easy to check). If $S$ is open, then $f^{-1}(S)$ is open in $\mathbb{R}^2$, thus Borel. So $\{\text{open sets}\} \subseteq \mathcal{A}$ and so the Borel algebra is a subset of $\mathcal{A}$. In particular, $A \in \mathcal{A}$.

Let $E = f^{-1}(A)$ which is a Borel set of $\mathbb{R}^2$. If $m(A) = 0$, let

$$E^y = \{x \in \mathbb{R} \mid (x, y) \in E\} = y + A$$

This is a null set since $m(y + A) = m(A) = 0$. Thus, $(m \times m)(E) = \int m(E^y) dm(y) = 0$. □

# 8 January 2016

**Problem 1.** *Let $E$ be a measurable subset of $[0, 1]$. Suppose there exists $\alpha \in (0, 1)$ such that*

$$m(E \cap J) \geq \alpha \cdot m(J)$$

*for all subintervals $J$ of $[0, 1]$. Prove that $m(E) = 1$.*

*Proof.* It's easy to see that $m(E) \leq 1$.

For any open $U \subseteq [0, 1]$, write $U = \sqcup_{i=1}^{\infty} I_i$ where each $I_i$ is an open interval. Then

$$m(E \cap U) = \sum_{i=1}^{\infty} m(E \cap I_i) \geq \sum_{i=1}^{\infty} \alpha m(I_i) = \alpha m(U).$$

Assume $m(E) < 1$, so $m(E^c) := a > 0$. We may find some open $U \supseteq E^c$ such that $m(U \cap E) = m(U \backslash E^c) < \epsilon$. So

$$\epsilon > m(U \cap E) \geq \alpha m(U) \geq \alpha m(E^c) = \alpha a > 0.$$

Letting $\epsilon \to 0$, this leads to a contradiction. □

**Problem 2.** *Let $f, g \in L^1([0, 1])$. Suppose*

$$\int_0^1 x^n f(x) dx = \int_0^1 x^n g(x) dx$$

*for all integers $n \geq 0$. Prove that $f(x) = g(x)$ a.e.*

*Proof.* See # 2 from January 2017. □

**Problem 3.** *Let $f, g \in L^1([0,1])$. Assume for all functions $\varphi \in C^\infty[0,1]$ with $\varphi(0) = \varphi(1)$ we have*

$$\int_0^1 f(t)\varphi'(t)dt = -\int_0^1 g(t)\varphi(t)dt.$$

*Show that $f$ is absolutely continuous and $f' = g$ a.e.*

*Proof.* Fix $x \in [0,1]$ and construct $h_n$ via

$$h_n(t) = \begin{cases} nt & t \in [0, 1/n] \\ 1 & t \in [1/n, x] \\ 1 - n(t-x) & t \in [x, x+1/n] \\ 0 & t \in [x+1/n, 1] \end{cases}$$

(i.e. $h_n(t)$ linearly connects the points $(0,0), (1/n, 1), (x, 1), (x + 1/n, 0)$, and $(1, 0)$.

Since $C^\infty[0,1]$ is dense in $\|\cdot\|_\infty$, we may use this example rather than some $\varphi \in C^\infty[0,1]$ (i.e. pass to the continuous case). Then

$$\int_0^1 f(t)h_n'(t)dt = \int_0^{1/n} f(t)ndt + 0 + \int_x^{x+1/n} f(t)(-n)dt + 0 \to f(0) - f(x)$$

where the limit follows from Lebesgue Differentiation Theorem. Also,

$$\begin{aligned}\int_0^1 g(t)h_n(t)dt &= \int_0^{1/n} nt\underbrace{g(t)}_{\to 0}dt + \int_{1/n}^x g(t)dt + \int_x^{x+1/n} g(t)dt - \int_x^{x+1/n} n\underbrace{(t-x)g(t)}_{\to 0 \text{ as } t\to x}dt + 0 \\ &\to 0 + \int_{1/n}^{x+1/n} g(t)dt - 0\end{aligned}$$

where the limit again follows from Lebesgue Differentiation Theorem. Taking the limit as $n \to \infty$ on both sides, we get $\int_0^x g(t)dt = \lim_n \int_{1/n}^{x+1/n} g(t)dt$. So

$$f(0) - f(x) = \lim_n \int_0^1 f(t)h_n'(t) = \lim_n -\int_0^1 g(t)h_n(t)dt = -\int_0^x g(t)dt$$

Implying $f(x) = f(0) + \int_0^x g(t)dt$. Then

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{f(0)+\int_0^{x+h} g(t)dt - f(0) - \int_0^x g(t)dt}{h} = \lim_{h\to 0} \frac{\int_x^{x+h} g(t)dt}{h} = g(x)$$

and $f$ is absolutely continuous. $\square$

**Problem 4.** *Let $\{g_n\}$ be a sequence of measureable functions on $[0,1]$ such that*

*(i) $|g_n(x)| \le C$, for a.e. $x \in [0,1]$*

*(ii) and $\lim_{n\to\infty} \int_0^a g_n(x)dx = 0$ for every $a \in [0,1]$.*

*Prove that for each $f \in L^1([0,1])$, we have*

$$\lim_{n\to\infty} \int_0^1 f(x)g_n(x)dx = 0.$$

*Proof.* Let $S = \text{span}\{\chi_{[0,a]} \mid a \in [0,1]\}$. Then $S$ is dense in the space of step functions in $L^1$. Step function space is dense in $L^1$ so $S$ is dense in $L^1$. Then for every $f \in L^1[0,1]$ there exists a sequence $h_m = \sum_{i=1}^{K_m} K_i^{(m)} \chi_{[0,a_i]} \to f$ in $L^1$.

For a fixed $m$,

$$\lim_n \int_0^1 h_m g_n dx = \sum_{i=1}^{K_m} K_i^{(m)} \lim_n \int_0^{a_i} g_n(x)dx = 0$$

where the second equality follows from (ii). For every $\epsilon > 0$, we can choose some $m$ such that $\|h_m - f\|_1 < \epsilon$.

For that $m$, choose soem $N$ such that $\left|\int_0^1 h_m g_n dx\right| < \epsilon$ for all $n > N$. Then

$$\begin{aligned} \left|\int_0^1 f(x)g_n(x)dx\right| &\le \left|\int_0^1 \big(f(x)-h_m(x)\big)g_n(x)dx\right| + \left|\int_0^1 h_m(x)g_n(x)dx\right| \\ &\le c\|f-h_m\|_1 + \epsilon \\ &< (c+1)\epsilon \end{aligned}$$

Thus, $\int_0^1 f(x)g_n(x)dx = 0$. $\square$

**Problem 5.** *(a) Let $X$ be a normed vector space and $Y$ be a closed linear subspace of $X$. Assume $Y$ is a proper subspace, that is, $Y \ne X$. Show that, for all $0 < \epsilon < 1$, there is an element $x \in X$ such that $\|x\| = 1$ and*

$$\inf_{y\in Y} \|x-y\| > 1-\epsilon$$

*Proof.* Fix some $x_0 \in X \backslash Y$, denote $\inf_{y \in Y} \|x_0 - y\| = d > 0$. Now for every $\epsilon > 0$ choose some $\delta > 0$ such that $\frac{d}{d+\delta} > 1 - \epsilon$.

Choose $y_0 \in Y$ such that $\|x_0 - y_0\| < d + \delta$. Let $x = \frac{x_0 - y_0}{\|x_0 - y_0\|}$ so $\|x\| = 1$ and

$$\inf_{y \in Y} \|x - y\| = \inf_{y \in Y} \left\| \frac{x_0 - y_0}{\|x_0 - y_0\|} - y \right\| = \frac{1}{\|x_0 - y_0\|} \inf_{y' \in Y} \|x_0 - y'\| > \frac{d}{d+\delta} > 1 - \epsilon.$$

□

(b) *Use part (a) to prove that, if $X$ is an infinite dimensional normed vector space, then the unit ball of $X$ is not compact.*

*Proof.* If we construct a sequence $\{x_n\}$ such that there are no convergent subsequences, we are done.

Assume we have chosen $\{x_1, x_2, \ldots, x_{n-1}\} \subseteq \overline{B_X}$. Let $Y = \text{span}\{x_1, x_2, \ldots, x_{n-1}\}$. By part (a), there exists some $x_n \in B$ such that $\|x_n\| = 1$ and $\inf_{y \in Y} \|x_n - y\| > 1/2$.

Then we have a sequence $\{x_n\} \subseteq \overline{B_X}$ such that $\|x_n - x_m\| > 1/2$ for all $n \neq m$ so no convergent subsequence may exist. □

**Problem 6.** *Let $\{f_k\}$ be a sequence of increasing functions on $[0, 1]$. Suppose*

$$\sum_{k=1}^{\infty} f_k(x)$$

*converges for all $x \in [0, 1]$. Denote the limit function by $f$, that is,*

$$f(x) = \sum_{k=1}^{\infty} f_k(x).$$

*Prove that*

$$f'(x) = \sum_{k=1}^{\infty} f_k'(x), \quad a.e.\ x \in [0, 1].$$

*Proof.* It's easy to see $f$ is increasing, so it's differentiable almost everywhere. Let $F_N = \sum_{n=1}^{N} f_n$ so $F_N \to f$ for all $x \in [0, 1]$. Choose an increasing sequence $N_k$ such taht $0 \leq f(1) - F_{N_k}(1) \leq 2^{-k}$. Then

$$\sum_{k=1}^{\infty} \big(f(1) - F_{N_k}(1)\big) \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Now, let $g(x) := \sum_{k=1}^{\infty} \big(f(x) - F_{N-k}(x)\big) = \sum_{k=1}^{\infty} \sum_{n=N_k+1}^{\infty} f_n(x)$.

Since $\sum_{n=N_k+1}^{\infty} f_n(x)$ is increasing as $x$ increases, then $g$ is increasing.

So $0 \leq g(x) \leq g(1) \leq 1$ and $g$ is differentiable almost everywhere. Now,

$$\frac{1}{h}\big(g(x+h)-g(x)\big) = \frac{1}{h}\sum_{k=1}^{\infty}\big(f(x+h)-F_{N_k}(x+h)\big)-\big(f(x)-F_{N_k}(x)\big).$$

So since $f'(x)-F_{N_k}(x) = \sum_{n=N_k+1}^{\infty} f_n(x)$ is increasing, $g'(x) \geq \sum_{k=1}^{\infty} f'(x)-F'_{N_k}(x) \geq 0$. Therefore, $\sum_{k=1}^{\infty} f'(x)-F_{N_k}(x)$ converges. So $\lim_k F'_{N_k}(x) = f'(x)$, implying $f'(x) = \sum_{k=1}^{\infty} f'_k(x)$ almost everywhere. □

**Problem 7.** *Suppose $f, g : [a,b] \to \mathbb{R}$ are both continuous and of bounded variation. Show that the set*

$$\{(f(t), g(t)) \in \mathbb{R}^2 \mid t \in [a,b]\}$$

*cannot cover the entire unit square $[0,1]\times[0,1]$.*

*Proof.* Define $r(t) = (f(t), g(t))$. Since $\mathbb{R}^2$ is finite dimensional, $\ell^1 \sim \ell^2$. Since $f$ and $g$ have bounded variation, so does $r$. Thus, we know that whenever $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ we have $\sum_{i=1}^{n} \|r(x_i)-r(x_{i-1})\|_2 < M$.

Now suppose $[0,1]\times[0,1]$ can be covered. Divide $[0,1]\times[0,1]$ into $n^2$ small squares with center $z_j$ and the length of each edge is $1/n$. Then choose $t_j$ such that $r(t_j) = z_j$.

Now relabel/reorder the $t_j$ in increasing order so that $s_1 < s_2 < \ldots < s_{n^2}$. Then since the distance between two centers is at least $1/n$,

$$\sum_{j=1}^{n^2-1} \|r(s_{j+1})-r(s_j)\|_2 \geq \sum_{j=1}^{n^2-1} 1/n = \frac{n^2-1}{n} \to \infty.$$

This is a contradiction! □

**Problem 8.** *Prove the following two statements:*

*(a) Suppose $f$ is a measurable function on $[0,1]$, then*

$$\|f\|_{L^\infty} = \lim_{p\to\infty} \|f\|_{L^p}$$

*Proof.* In $[0,1]$, by Hölder, we know that $\|f\|_p \leq \|f\|_q$ when $p \leq q$. Also, $\|f\|_p \leq \|f\|_\infty$ for all $p$. Therefore, $\|f\|_p \nearrow \leq \|f\|_\infty$ and so $\lim_p \|f\|_p \leq \|f\|_\infty$.

On the other hand, for every $\epsilon > 0$, let $E = \{x \mid |f(x)| > \|f\|_\infty - \epsilon\}$ and $0 < \mu(E) \leq 1$ since $\|f\|_\infty = \text{esssup}\,|f(x)|$. Then $\|f\|_p^p \geq \int_E |f|^p > \big(\|f\|_\infty - \epsilon\big)^p \mu(E)$. Take $p \to \infty$ so $\lim_p \|f\|_p \geq \|f\|_\infty - \epsilon$, implying $\lim_p \|f\|_p \geq \|f\|_\infty$. □

*(b) If $f_n \geq 0$ and $f_n \to f$ in measure, then $\int f \leq \liminf \int f_n$.*

*Proof.* Choose a subsequence $\{f_{n_k}\}$ such that $\lim_k \int f_{n_k} = \liminf \int f_n$. Since $f_n \to f$ in measure, $f_{n_k} \to f$ in measure, so there exists a further subsequence $\{f_{n_{k_\ell}}\} \to f$ a.e. Then by Fatou's Lemma,

$$\int f = \int \lim_\ell f_{n_{k_\ell}} \leq \liminf_\ell \int f_{n_{k_\ell}} = \liminf_k f_{n_k} = \liminf_n \int f_n.$$

□

**Problem 9.** *Suppose $\{f_n\}$ is a sequence of functions in $L^2([0,1])$ such that $\|f_n\|_{L^2} \leq 1$. If $f$ is measurable and $f_n \to f$ in measure, then*

*(a) $f \in L^2([0,1])$;*

*Proof.* $f_n \to f$ in measure implies $\{f_{n_k}\} \to f$ almost everwhere which implies $|f_{n_k}|^2 \to |f|^2$ almost everywhere. By Fatou's Lemma,

$$\int_0^1 |f|^2 dx \leq \lim_n \int_0^1 |f_{n_k}|^2 dx \leq 1.$$

So $f \in L^2$. □

*(b) $f_n \to f$ weakly in $L^2$;*

*Proof.* Let $g \in L^2$. We want to show that $f_n g \to fg$ in $L^1$. Now, $f_n \to f$ in measure, then $f_n g \to fg$ in measure and thus is Cauchy in measure.

Define $A_{m,n} = \{x \in [0,1] \mid |f_n g(x) - f_m g(x)| \geq \epsilon\}$. Then

$$\int_0^1 |f_n g - f_m g| dx = \int_{A_{m,n}} |f_n g(x) - f_m g(x)| dx + \int_{[0,1]\setminus A_{m,n}} |f_n g(x) - f_m g(x)| dx \leq \int_{A_{m,n}} |f_n g| + |f_m g| dx + \epsilon.$$

We know for all $\epsilon > 0$ there exists some $\delta > 0$ such that $\mu(A_{m,n}) < \delta$,

$$\int_{A_{m,n}} |f_n g| \leq \left(\int_{A_{m,n}} |f_n|^2 dx\right)^{1/2} \left(\int_{A_{m,n}} |g|^2 dx\right)^{1/2} \leq \left(\int_{A_{m,n}} |g|^2\right)^{1/2} < \epsilon.$$

since $g \in L^2$. Then since $\{f_n g\}$ is Cauchy in measure, there exists some $N$ such that for all $m, n > N$, $\mu(A_{m,n}) < \delta$. Then $\int_0^1 |f_n g - f_m g| dx < 3\epsilon$ implies $\{f_n g\}$ is Cauchy in $L^1$.

Therefore, there exists some $h \in L^1$ such taht $f_n g \to h$ in $L^1$.

We know $f_n g \to fg$ in measure, so $f_{n_k} g \to fg$ almost everywhere. Also, $\forall \epsilon > 0$, $\exists \delta > 0$ such that $\int_A |f_{n_k} g| < \epsilon$ for all $A$ such that $\mu(A) < \delta$.

Therefore, $\{f_{n_k}\}$ is uniformly integrable. By Viteli Convergence Theorem, $f_{n_k} g \to fg$ in $L^1$. Thus, $h = fg$ so $f_n g \to fg$ in $L^1$. So $f_n \to f$ weakly.

Note: We could also have used the uniqueness of limit in the measure. □

*(c) $f_n \to f$ with respect to norm in $L^p$ for $1 \leq p < 2$.*

*Proof.* Define $E_n = \{x \mid |f_n(x) - f(x)| \geq \epsilon\}$. From problem 8 on this exam, we know $\|f_n\|_p \leq \|f_n\|_2 \leq 1$ and $\|f\|_p \leq \|f\|_2 < \infty$. Then

$$\int |f_n - f|^p = \int_{E_n} |f_n - f|^p + \int_{E_n^c} |f_n - f|^p dx \leq 2^{p-1} \int_{E_n} |f_n|^p + |f|^p + \epsilon$$

where the inequality follows from the fact that $|a-b|^p \leq 2^{p-1}(|a|^p + |b|^p)$.

Since $f_n \to f$ in measure and $m(E_n) \to 0$ as $n \to \infty$, so since $f \in L^p$ then $\int_{E_n} |f|^p dx \to 0$ as $n \to \infty$.

For some $A \subseteq [0,1]$, we have

$$\int_A |f_n|^p = \int_0^1 |f_n|^p \chi_A \leq \| |f_n|^p \|_{2/p} \|\chi_A\|_{2/2-p} = \|f_n\|_2^p m(A)^{\frac{2}{2-p}} \leq m(A)^{\frac{2}{2-p}}.$$

So similar to the previous case, we can take $m(E_n)$ small enough such that $\int_{E_n} |f_n|^p dx < \epsilon$ for any fixed $1 \leq p < 2$. □

*There are a few hints in the qual*

**Problem 10.** *Suppose $E$ is a measurable subset of $[0,1]$ with Lebesgue measure $m(E) = \frac{99}{100}$.*

*Show that there exists a number $x \in [0,1]$ such that for all $r \in (0,1)$,*

$$m(E \cap (x-r, x+r)) \geq \frac{r}{4}.$$

*Hint: Use the Hardy-Littlewood maximal inequality*

$$m(\{x \in \mathbb{R} \mid Mf(x) \geq \alpha\}) \leq \frac{3}{\alpha}\|f\|_1$$

*for all $f \in L^1(\mathbb{R})$. Here $Mf$ denotes the Hardy-Littlewood Maximal function of $f$.*

*Proof.* The Hardy-Littlewood Maximal function of $\chi_A$ is

$$M\chi_A = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \chi_A(x) dx = \sup_{r>0} \frac{1}{2r} m(A \cap (x-r, x+r)).$$

Assume the result is not true. Then $\forall x \in [0,1]$, $\exists r_x \in (0,1)$ such that $m(E \cap (x - x_r, x + x_r)) < \frac{r_x}{4}$. This happens if and only if $\frac{1}{2r_x} m(E \cap (x - r_x, x + r_x)) < 1/8$ which is equivalent to $\frac{1}{2r_x} m(E^c \cap (x - r_x, x + r_x)) \geq \frac{7}{8}$.

Now set $A = E^c$ so $M\chi_A(x) \geq \frac{7}{8}$. However,

$$m\big(\{x \in [0,1] \mid M\chi_A(x) \geq \frac{7}{8}\}\big) \leq 3\frac{8}{7}\|\chi_A\| = \frac{24}{7}\frac{1}{100} = \frac{24}{100}.$$

But we need it to be equal to 1. Contradiction! □

# 9 August 2015

**Problem 1.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. For each $t \in \mathbb{R}$ define*

$$f_t(x) = f(t+x), \quad x \in \mathbb{R}.$$

*Prove that $f_t(x)$ is a Borel measurable function (in $x$) for each fixed $t \in \mathbb{R}$.*

*Proof.* We see that

$$f_t^{-1}(-\infty, a) = \{x \mid f(x+t) \in (-\infty, a)\} = \{x \mid x+t \in f^{-1}(-\infty, a)\} = f^{-1}((-\infty, a)) - t = B - t.$$

Since $T_t(x) = x + t$ is continuous, then $T_t^{-1}(B) = B - t$ is Borel. □

**Problem 2.** *Justify the statement that*

$$\int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dx\ dy = \int_0^1 \int_0^1 \frac{(x-y)\sin(xy)}{x^2+y^2} dy\ dx.$$

*Proof.* We just need to show that $\int_0^1 \int_0^1 \left| \frac{(x-y)\sin(xy)}{x^2+y^2} \right| dxdy < \infty$. But

$$\int_0^1 \int_0^1 \left| \frac{(x-y)\sin(xy)}{x^2+y^2} \right| dxdy = \int_0^{\pi/2} \int_0^{\sqrt{2}} \left| \frac{r\cos\theta - r\sin\theta}{r^2} \right| |r| dr d\theta \le 2 \int_0^{\pi/2} \int_0^{\sqrt{2}} dr d\theta = \sqrt{2}\pi < \infty.$$

So the function is in $L^1$ and Fubini gives us the desired result. □

**Problem 3.** *Assume that $(f_n)$ is a sequence in $C[0,1]$.*

(a) *Show that $(f_n)$ converges weakly to 0 if and only if $(f_n)$ is bounded in $C[0,1]$ and $\lim_{n\to\infty} f_n(t) = 0$ for all $t \in [0,1]$.*

*Proof.* $\Rightarrow$) We know $C[0,1]^* = \mathcal{M}[0,1]$. Then $f_n \to 0$ weakly implies $\int f_n d\mu \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Choose $\mu = \delta_t$ so

$$\int f_n d\delta_t = f_n(t) \to 0 \quad \forall t \in [0,1]$$

(this follows from the fact that weak convergence implies uniformly bounded). Consider

$$\begin{aligned} \chi : C[0,1] &\to C[0,1]^{**} = \mathcal{M}[0,1]^* \\ \chi(f_n)(\mu) &= \mu(f_n) \end{aligned}$$

Since $\mu(f_n) \to 0$ then $\chi(f_n)(\mu) \to 0$ for all $\mu \in \mathcal{M}[0,1]$. Since convergent sequences are bounded, then $\sup_n |\chi(f_n)(\mu)| \le M$.

By the uniform boundedness theorem, $\sup_n \|\chi(f_n)\| < \infty$. By isometry, $\|f_n\| = \|\chi(f_n)\|$ so $\sup_n \|f_n\| < \infty$.

$\Leftarrow$) By Dominated Convergence Theorem, $f_n \to 0$ in $L^1(\mu)$. So therefore, $|\int f_n d\mu| \le \int |f_n| d|\mu| \to 0$. So $f_n \to 0$ weakly. □

(b) *Show that if $(f_n)$ converges weakly in $C[0,1]$, then it converges in norm in $L_p[0,1]$ for all $1 \le p < \infty$.*

*Proof.* WLOG $f_n \to 0$ weakly. By (a) we know $f_n(t) \to 0$ and $\|f_n\|_\infty$ is bounded. Thus, $|f_n(t)|^p \to 0$ pointwise and $\||f_n|\|_\infty$ is bounded.

By the Dominated Convergence Theorem, we have $\|f_n\|_p \to 0$. □

**Problem 4.** *Let $A$ be a Lebesgue null set in $\mathbb{R}$. Prove that*

$$B := \{e^x \mid x \in A\}$$

*is also a null set.*

*Proof.* First, assume $A \subseteq [0,1]$. Then $f(x) = e^x$ is Lipschitz-continuous (i.e. $|f(x) - f(y)| \le M|x-y|$ for some $M$). Since $m(A) = 0$, we can find $\cup_{k=1}^\infty B_k$ where $B_k$ are open intervals such that $A \subseteq \cup_{k=1}^\infty B_k$ and $m\left(\cup_{k=1}^\infty B_k\right) < \epsilon$. Then

$$m(f(A)) \le m\left(f\left(\cup_{k=1}^\infty B_k\right)\right) \le \sum_{k=1}^\infty M m(B_k) < M\epsilon.$$

So $m(f(A)) = 0$. Now we can write $A = \cup_{n=-\infty}^\infty A \cap [n, n+1]$ so $m(f(A)) = \sum_{-\infty}^\infty m(f(A \cap [n, n+1])) = 0$. □

**Problem 5.** *(a) Define absolute continuity of a function $f : \mathbb{R} \to \mathbb{R}$ and of a function $f : [a,b] \to \mathbb{R}$.*

*Proof.* The function $f : [a,b] \to \mathbb{R}$ is absolutely continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ such that whenever a finite sequence of disjoint subintervals $(x_k, y_k) \subseteq I$ satisfies $\sum_{k=1}^N (y_k - x_k) < \delta$ then $\sum_{k=1}^\infty |f(y_k) - f(x_k)| < \epsilon$. □

(b) *Show that if $f$ and $g$ are absolutely continuous on $[a,b]$, $a, b \in \mathbb{R}$, $a < b$, then $f \cdot g$ is absolutely continuous on $[a,b]$.*

*Proof.* Since $f$ and $g$ are continuous on $[a,b]$, then they achieve a maximum so we can let $M_f = \sup\{f(x) \mid a \le x \le b\} < \infty$, $M_g = \sup\{g(x) \mid a \le x \le b\}$.

Fix $\epsilon > 0$. Then there exists some $\delta_f, \delta_g > 0$ such that

$$\sum (y_k - x_k) < \delta_f \quad \Rightarrow \quad \sum |f(y_k) - f(x_k)| < \frac{\epsilon}{2M_g}$$
$$\sum (y_k - x_k) < \delta_g \quad \Rightarrow \quad \sum |f(y_k) - f(x_k)| < \frac{\epsilon}{2M_f}$$

Choose finite and disjoint such that $\sum y_k - x_k < \min(\delta_f, \delta_g)$. Then

$$\begin{aligned} \sum |f(y_k)g(y_k) - f(x_k)g(x_k)| &\le \sum |f(y_k)g(y_k) - f(y_k)g(x_k)| + \sum |f(y_k)g(x_k) - f(x_k)g(x_k)| \\ &\le \sum |f(y_k)||g(y_k) - g(x_k)| + \sum |g(x_k)||f(y_k) - f(x_k)| \\ &\le M_f \sum |g(y_k) - g(x_k)| + M_g \sum |f(y_k) - f(x_k)| \\ &\le M_f \frac{\epsilon}{2M_f} + M_g \frac{\epsilon}{2M_g} \\ &= \epsilon \end{aligned}$$

This is what we wanted. □

(c) *Give an example to show that (b) is false if $[a, b]$ is replaced by $\mathbb{R}$.*

*Proof.* Take $f(x) = g(x) = x$ so $fg = x^2$. Then

$$|(x + \delta)^2 - x^2| = |x^2 + 2\delta x + \delta^2 - x^2| = |2\delta x + \delta^2| \to \infty \quad \text{as } x \to \infty.$$

So there does not exist any $\delta$ such that $|fg(y) - fg(x)| < \epsilon$ (even for just one interval!) □

**Problem 6.** *Let $X$ and $Y$ be Banach spaces and $T : X \to Y$ be a one-to-one, bounded and linear operator for which the range $T(X)$ is closed in $Y$. Show that for each continuous linear functional $\phi$ on $X$ there is a continuous linear functional $\psi$ on $Y$, so that $\phi = \psi \circ T$.*

*Proof.* Since $T : X \to T(X)$ is bijective, by teh open mapping theorem, $T^{-1}$ is bounded so $\phi \circ T^{-1} \in T(X)^*$.

Then by the Hahn-Banach, there exists some $\psi \in Y^*$ such that $\psi(y) = (\phi \circ T^{-1})(y)$ for all $y \in Y$.

For any $x \in X$, $T(x) = y \in Y$ and we have

$$\phi(x) = \phi\big(T^{-1}(Tx)\big) = \psi(Tx) = (\psi \circ T)(x).$$

Since this is true for all $x \in X$, $\phi = \psi \circ T$. □

**Problem 7.** *State the Open Mapping Theorema nd the Closed Graph Theorem for Banach spaces. Derive the Open Mapping Theorem from the Closed Graph Theorem.*

*Proof.* Assume $T : X \to Y$ is surjective, linear, and bounded. WLOG we want to show $B(0, \delta) \subseteq T(B(0, 1))$ for some $\delta > 0$. Define

$$\begin{aligned} G : Y &\to X/\ker(T) \\ y &\mapsto [x] = x + \ker(T) \quad \text{where } y = Tx \end{aligned}$$

Then $G$ is well-defined, because $T$ is surjective.

Claim: $G$ is closed.

Assume $y_n \to y$ in $Y$ and $G(y_n) \to [x]$ in $X/\ker(T)$. WTS $G(y) = [x] \Leftrightarrow Tx = y$.

We have $Tx_n = y_n$ so since $[x_n] \to [x]$ then $\|[x_n] - [x]\| = \inf_{z \in \ker T} \|x_n - x - z\| \to 0$. Then take $(z_n) \subseteq \ker(T)$ such that $\|x_n - x - z_n\| < 1/n$. So $x_n - z_n \to x$. Then

$$\|T(x_n - z_n) - T(x)\| \leq \|T\| \|x_n - x - z_n\| \to 0.$$

Thus, $T(x_n - z_n) = T(x_n) \to T(x)$. And also $T(x_n - z_n) = T(x_n) = y_n \to y$. Together, these imply $T(x) = y$. So $G$ is closed, and the claim holds.

By the closed graph theorem, $G$ is bounded so there exists some $\delta > 0$ such that $G(B(0, \delta)) \subseteq B(0, 1)$ in $X/\ker(T)$. Now, let $y \in B(0, \delta)$ so then $[x] = G(y) \in B(0, 1)$. Thus, if $\inf_{z \in \ker(T)} \|x - z\| < 1$, then there exists some $z_0 \in \ker(T)$ such that $\|x - z_0\| < 1$. This implies $y = Tx = T(x - z_0) \in T(B(0, 1))$ so $B(0, \delta) \subseteq T(B(0, 1))$. □

**Problem 8.** *Let $Y$ be a closed subspace of a Banach space $X$, with norm $\|\cdot\|$. Let $\|\cdot\|_1$ be a norm on $Y$ which is equivalent to $\|\cdot\|$, meaning that there is a $C \geq 1$ so that*

$$\frac{1}{C}\|y\|_1 \leq \|y\| \leq C\|y\|_1 \text{ for all } y \in Y.$$

*Let $S$ be the set of all linear functions $\phi : X \to \mathbb{R}$, so that*

*(i) $|\phi(y)| \leq \|y\|_1$ for all $y \in Y$, and*

*(ii) $|\phi(x)| \leq C\|x\|$ for all $x \in X$.*

*Prove the following statements*

*(a) $\|x\|_2 := \sup_{\phi \in S} |\phi(x)|$, $x \in X$, defines a norm on $X$.*

*Proof.* Easy to check. □

*(b) $\|y\|_2 = \|y\|_1$ for $y \in Y$.*

*Proof.* Since $|\phi(y)| \leq \|y\|_1$ then $\|y\|_2 \leq \|y\|_1$.

On the other hand, from the Hahn-Banach separation theorem, for all $y \neq 0$, there exists some $\phi \in X^*$ such that $\|\phi\| = 1$ and $\phi(y) = \|y\|_1$ so $\|y\|_2 \geq \|y\|_1$.

To check that $\phi \in S$: $|\phi(y)| = \|y\|_1$ and $|\phi(x)| \leq \|\phi\|\|x\| = \|x\|$. □

*(c) The norms $\|\cdot\|_2$ and $\|\cdot\|$ are equivalent on $X$.*

*Proof.* We just need to consider this on $X\backslash Y$. For $x \in X\backslash Y$, we have

$$\|x\|_2 = \sup_{\phi \in S} |\phi(x)| \leq C\|x\|.$$

Again by Hahn-Banach, for $\tilde{x} \neq 0$, there exists some $\phi \in X^*$ such that $\phi(\tilde{x}) = \|\tilde{x}\|$ and $\|\phi\| = 1$. Define $\psi = \frac{1}{C}\phi$ so $\|\psi\| = \frac{1}{C}$.
Then to see $\psi \in S$:

- $|\psi(x)| \leq \frac{1}{C}\|x\| \leq C\|x\|$ for all $x \in X$
- $|\psi(y)| \leq \frac{1}{C}\|y\| \leq \|y\|_1$ for all $y \in Y$

So $\psi \in S$ and $\|\tilde{x}\| \geq |\psi(\tilde{x})| = \frac{1}{C}\|\tilde{x}\|$ so $\frac{1}{C}\|x\| \leq \|x\|_2 \leq C\|x\|$. □

**Problem 9.** *Let $f$ be increasing on $[0,1]$ and let*

$$g(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}, \qquad \textit{for } 0 < x < 1.$$

*Prove that if $A = \{x \in (0,1) \mid g(x) > 1\}$ then*

$$f(1) - f(0) \geq m^*(A).$$

*Proof.* For $x \in A$,

$$\limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} > 1$$

so for all $\epsilon > 0$, there exists some $h > 0$ such that $2h < \epsilon$ and $\frac{f(x+h)-f(x-h)}{2h} > 1$ if and only if $f(x+h) - f(x-h) > 2h$.

Let $I = \{(x-h, x+h) \mid x \in A, 2h < \epsilon, (x-h, x+h) \subseteq [0,1]\}$. Then $I$ covers $A$ in the sense of Vitali. By Vitali's Lemma, for every $\epsilon > 0$, there exists $I_1, I_2, \ldots, I_n$ disjoint from $I$ such that $m^*\left(A \backslash \cup_{i=1}^n I_i\right) < \epsilon$.

Since $m^*(A) = m^*\left(A \backslash \cup_{i=1}^n I_i\right) + m^*\left(\cup_{i=1}^n I_i\right)$ for all $I_i$ then write $I_i = (x_i - h_i, x_i + h_i)$ and $x_1 - h_1 < x_1 + h_1 < x_2 - h_2 < \ldots < x_n + h_n$. Then

$$m^*(A) < \epsilon + \sum_{i=1}^n 2h_i < \epsilon + \sum_{i=1}^n |f(x_i + h_i) - f(x_i - h_i)|.$$

Since $f$ is increasing, $\sum_{i=1}^n \big(f(x_i + h_i) - f(x_i - h_i)\big) \leq f(1) - f(0)$.

So $m^*(A) < \epsilon + f(1) - f(0)$ so $m^*(A) \leq f(1) - f(0)$. □

**Problem 10.** *(a) State a version of the Stone-Weierstrass Theorem.*

*Proof.* See textbook. □

*(b) Let $A$ be a uniformly dense subspace of $C[0,1]$ and let*

$$B = \left\{ F(x) \mid F(x) = \int_0^x f(t)dt, \quad 0 \le x \le 1, f \in A \right\}.$$

*Prove that $B$ is uniformly dense in*

$$C_0[0,1] := \{g \in C[0,1] \mid g(0) = 0\}.$$

*Proof.* Define $B' = \{F(x) \mid F(x) = \int_0^x f(t)dt, 0 \le x \le 1, f \in C[0,1]\}$. First show $B$ is dense in $B'$.

For every $F \in B'$, $G = B$, $F(x) = \int_0^x f(t)dt$ and $G(x) = \int_0^x g(t)dt$. Then

$$\|F(x) - G(x)\|_\infty \le \int_0^1 |f - g|dt \le \|f - g\|_\infty.$$

Since $A$ is dense in $C[0,1]$, then $\|f - g\|_\infty < \epsilon$ so $\|F - G\|_\infty < \epsilon$. So $B$ is indeed dense in $B'$.

Then we will show $B'$ is an algebra (in order to use Stone-Weierstrass). Let $F, G \in B'$ so

$$F(x)G(x) = \int_0^x f(t)dt \int_0^x g(s)ds = \int_0^x \int_0^x f(t)g(s)dsdt = \int_0^x F(t)g(t) + G(t)f(t)dt = \int_0^x \int_0^t f(s)g(t) + g(s)f(t)dsdt.$$

Since $F(t)g(t) + G(t)f(t) \in C[0,1]$ then $FG \in B'$.

Also, $x = \int_0^1 1dt \in B'$ so $B'$ separates points.

By Stone-Weierstrass, $B'$ is dense in $C_0[0,1]$ since any function $F \in B'$, $F(0) = 0$. So $B$ is dense in $C_0[0,1]$. □

*(c) Prove that the span of $\{\sin(nx) \mid n \in \mathbb{N}\}$ is dense in $C_0[0,1]$.*

*Proof.* $\sin(nx) = \int_0^x n\cos(nx)dt$. From part (b), it is sufficient to show

$$A = \text{span}\{n\cos(nx)\} = \text{span}\{\cos(nt)\}$$

is dense in $C[0,1]$. $A$ is an algebra:

- $\cos(nt)\cos(mt) = \frac{1}{2}\big(\cos((m+n)t) + \cos((m-n)t)\big) \in A$
- $\cos(t)$ separates $[0,1]$ (since $1 < \pi/2$) so $A$ is dense in $C[0,1]$.

□

# 10 January 2015

**Problem 1.** *Let $f \in L^1(\mathbb{R})$. If*

$$\int_a^b f(x)dx = 0$$

*for all rational numbers $a < b$, prove that $f(x) = 0$ for almost all $x \in \mathbb{R}$.*

*Proof.* Let $E^+ := \{x \mid f(x) > 0\}$. Assume $m(E^+) > 0$ (the same argument will show $E^- := \{x \mid f(x) < 0\}$ has measure zero).

There exists some $n$ such that $E^+ \cap [n, n+1]$ has positive measure. Consider $F$ closed in $\mathbb{R}$ and $F \subseteq E^+ \cap [n, n+1]$ with $m(F) > 0$. Then $[n, n+1] \backslash F$ is open in $[n, n+1]$. Thus, $[n, n+1] \backslash F = \cup_{n=1}^{\infty} I_n$ for $I_n$ being disjoint open intervals in $[n, n+1]$.

For all $I_n = (a_n, b_n)$, there exists some $(a_{n_i})_i$, $(b_{n_i})_i \subseteq \mathbb{Q}$ such that $a_{n_i} \to a_n$ and $b_{n_i} \to b_n$. Since

$$\int_{a_n}^{b_n} f(x)dx = \int_{\mathbb{R}} f(x)\chi_{[a_n,b_n]}dx \qquad \lim_i f(x)\chi_{[a_{n_i},b_{n_i}]} = f(x)\chi_{[a_n,b_n]}$$

then $|f(x)\chi_{[a_{n_i},b_{n_i}]} \le |f(x)| \in L^1$ so by Dominated Convergence Theorem, $\int_{a_n}^{b_n} f(x)dx = 0$.

Since $\int_F f(x)dx > 0$, by condition we know $\int_n^{n+1} f(x)dx = 0$ for all $n$. So then $\int_{[n,n+1]\backslash F} f(x)dx < 0$.

So there exists some $I_m = (a_m, b_m)$ such that $\int_{I_m} f(x)dx < 0$. Contradiction!

*Proof #2 as in Problem 3 from August 2014, not restricted to rationals with $f \in L^1$.*

*For every open $U$, write $U = \cup_{n=1}^{\infty}(a_n, b_n)$ for disjoint open intervals, so*

$$\int_U f(x)dx = \sum_{n=1}^{\infty} \int_{a_n}^{b_n} f(x)dx = 0.$$

*For every compact $K \subseteq (a, b)$ then $(a, b)\backslash K$ is open in $\mathbb{R}$ and*

$$\int_K f(x)dx = \int_a^b f(x)dx - \int_{(a,b)\backslash K} f(x)dx$$

*(because each is finite). Suppose $E^+ = \{x \mid f(x) > 0\}$ has positive measure. Since $E^+ = \cup_n E_n$ where $E_n = \{x \mid f(x) > 1/n\}$ so there must exist some $n$ such that $m(E_n) > 0$.*

*By inner regularity, there exists some $K \subseteq E_n$ with $m(K) > 0$. Then*

$$0 = \int_K f(x)dx > \int_K \frac{1}{n}dx = \frac{1}{n}m(K) > 0$$

*Contradiction!* □

**Problem 2.** *Let* $\{g_n\}_{n=1}^{\infty}$ *and* $g$ *be in* $L^1(\mathbb{R})$ *and satisfy*

$$\lim_{n\to\infty} \|g_n - g\|_1 = 0.$$

*Prove that there is a subsequence of* $\{g_n\}_{n=1}^{\infty}$ *that converges pointwise almost everywhere to* $g$.

*Proof.* Step 1: Suppose $g_n \to g$ in $L^1$. Let $E_{n,\epsilon} = \{x \mid |f_n(x) - f(x)| \geq \epsilon\}$. Then

$$\int |f_n - f| \geq \int_{E_{n,\epsilon}} |f_n - f| \geq \epsilon\mu(E_{n,\epsilon})$$

So then $\mu(E_{n,\epsilon}) \leq \frac{1}{\epsilon}\int |f_n - f| \to 0$.

Step 2: We will show that if $g_n \to g$ in measure, then there exists a subsequence that converges to $g$ pointwise almost everywhere.

Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{g_{n_k}\}$ such that if

$$E_j = \{x \mid |g_{n_j}(x) - g_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \cup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \cap_k F_k$ so $\mu(F) = 0$.

For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|g_{n_i}(x) - g_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |g_{n_\ell}(x) - g_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So $g_{n_k}$ is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim g_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $g_{n_k} \to f$ almost everywhere and $g_n \to f$ in measure since

$$\mu(\{x \mid |g_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |g_n(x) - g_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |g_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x) - g(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - g_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |g_n(x) - g(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = g$ almost everywhere. Thus, $\{g_{n_k}\}$ converges to $g$ almost everywhere. $\square$

**Problem 3.** *Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\mathcal{N} \subseteq \mathcal{M}$ be a $\sigma$-algebra. If $f \geq 0$ is $\mathcal{M}$-measurable and $\mu$-integrable then prove that there exists an $\mathcal{N}$-measurable and $\mu$-integrable function $g \geq 0$ so that*

$$\int_E g d\mu = \int_E f d\mu, \quad E \in \mathcal{N}.$$

*Proof.* Define $\nu(E) = \int_E f d\mu$ a finite positive measure on $(X, \mathcal{N}, \mu)$. Then since $\mu(E) = 0$, $\nu(E) = 0$ so $\nu \ll \mu$.

Then by Radon-Nikodym Theorem, there exists some $g : X \to [0, \infty)$ and $\mathcal{N}$-measurable and $g \in L^1(\mu)$ such that $\nu(E) = \int_E g d\mu$. Then

$$\nu(E) = \int_E f d\mu = \int_E g d\mu \quad \forall E \in \mathcal{N}.$$

<u>Note:</u> Folland doesn't mention positive but there are other versions that give positive. $\square$

**Problem 4.** *(a) State the closed graph theorem.*

*Proof.* See wikipedia. $\square$

*(b) If $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is a linear operator satisfying*

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad x, y \in \mathcal{H},$$

*then prove that $T$ is bounded.*

*Proof.* Let $x_n \to x$ and $Tx_n \to y$. We want to show $Tx = y$.

$$\underbrace{\langle Tx_n, z \rangle}_{\to \langle y, z \rangle} = \langle x_n, Tz \rangle \to \langle x, Tz \rangle = \langle Tx, z \rangle.$$

So $\langle Tx - y, z \rangle = 0$ for all $z \in \mathcal{H}$ so then $Tx - y = 0$ and so $Tx = y$. $\square$

**Problem 5.** *Let $f, g \in L^1(\mathbb{R})$. Prove that $h \in L^1(\mathbb{R})$, where $h(x)$ is defined by*

$$h(x) = \int_{\mathbb{R}} f(y) g(x - y) dy$$

*whenever this integral is finite.*

*Proof.* We want to show that $\int_{\mathbb{R}} |h(x)| dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y) g(x - y) dy \right| dx < \infty$. Indeed,

$$\begin{aligned}\int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(y)g(x-y)dy \right| dx &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)||g(x-y)|dydx \\ &= \int_{\mathbb{R}} |f(y)| \left( \int_{\mathbb{R}} |g(x-y)|dx \right) dy \\ &= \int_{\mathbb{R}} |f(y)| \|g\|_1 dy \\ &= \|g\|_1 \int_{\mathbb{R}} |f(y)|dy \\ &= \|g\|_1 \|f\|_1 < \infty.\end{aligned}$$

□

**Problem 6.** *Let $f, g \in C[0,1]$ with $f(x) < g(x)$ for all $x \in [0,1]$.*

*(a) Prove that there is a polynomial $p(x)$ so that*

$$f(x) < p(x) < g(x), \quad x \in [0,1].$$

*Proof.* Let $\epsilon = \inf\{g(x) - f(x) \mid x \in [0,1]\}$. Since $h(x) = g(x) - f(x) > 0$ on $[0,1]$ and attains a minimum on the compact set $[0,1]$ then the inf is attained and thus is positive. So $\epsilon > 0$.

By Stone-Weierstrass, polynomials are dense in $\mathcal{C}[0,1]$ so there exists a polynomial $p(x)$ such that $\left\| p - \left(\frac{f+g}{2}\right) \right\|_\infty < \epsilon/2$. Then

$$p(x) < \frac{f(x)+g(x)}{2} + \frac{\epsilon}{2} \leq \frac{1}{2}\big(f(x) + g(x) + (g(x) - f(x))\big) = \frac{1}{2}(2g(x)) = g(x).$$

$$p(x) > \frac{f(x)+g(x)}{2} - \frac{\epsilon}{2} > \frac{1}{2}\big(f(x) + g(x) - (g(x) - f(x))\big) = \frac{1}{2}(2f(x)) = f(x).$$

So $f(x) < p(x) < g(x)$.

Remark: Let $M = \max\{g(x) - f(x)\}$ then

$$|g(x) - f(x)| \leq \left| g(x) - \left(\frac{f+g}{2}\right)(x) \right| + \left| \left(\frac{f+g}{2}\right)(x) - p(x) \right| < \frac{M}{2} + \epsilon.$$

*Alternative Proof.* Let $M := \min_{x\in[0,1]} g(x) - f(x)$. By Stone-Weierstrass, there exists some $\tilde{p}(x)$ polynomial such that $\|\tilde{p}(x) - g(x)\|_\infty < M/3$. Let $p(x) = \tilde{p}(x) - M/2$. Then

$$g(x) - p(x) = g(x) - \tilde{p}(x) + \frac{M}{2} > \frac{-M}{3} + \frac{M}{2} = \frac{M}{6} > 0.$$

$$p(x) - f(x) = p(x) - g(x) + g(x) - f(x) = \tilde{p}(x) - g(x) - \frac{M}{2} + \big(g(x) - f(x)\big) > \frac{-M}{3} - \frac{M}{2} + M = \frac{M}{6} > 0.$$

So $f(x) < p(x) < g(x)$. □

*(b) Prove that there is an increasing sequence of polynomial* $\{p_n(x)\}_{n=1}^{\infty}$ *so that*

$$f(x) < p_n(x) < g(x), \quad x \in [0,1],$$

*and* $p_n \to g$ *uniformly on* $[0,1]$.

*Proof.* Find $p_1$ such taht $g - \frac{1}{2} < p_1 < g$ with $\left\|p_1 - \left(\frac{g+g-1}{2}\right)\right\| < \frac{1}{4}$. Then $|g(x) - p_1(x)| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Recursively find a polynomial $p_n$ such that $p_{n-1} < p_n < g$ with $\left\|p_n - \left(\frac{g+p_{n-1}}{2}\right)\right\| < \frac{1}{2^{n-1}}$, implying

$$|g(x) - p_n(x)| < \frac{M_{n-1}}{2} + \frac{1}{2^{n+1}} = \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}.$$

So $M_n < \frac{1}{2^n}$. Then for every $\epsilon > 0$ choose $N$ such that $\frac{1}{2^N} < \epsilon$, so for all $n > N$

$$|p_n(x) - g(x)| < \frac{1}{2^N} < \epsilon \qquad \forall x \in [0,1].$$

*Alternative Proof.* From part (a) we can find $f(x) < p_1(x) < g(x)$. Repeating, we can find $p_1(x) < p_2(x) < g(x)$. By requiring $\epsilon_n$ instead of $M$, in $\|\tilde{p}(x) - g(x)\|_\infty < \epsilon$ and letting $\epsilon_n \to 0$, we get

$$\|p_n(x) - g(x)\|_\infty \leq \|p_n(x) - \tilde{p_n}(x)\|_\infty + \|\tilde{p_n}(x) - g\|_\infty < \frac{\epsilon_n}{2} + \frac{\epsilon_n}{3} = \frac{5}{6}\epsilon_n \to 0.$$

□

**Problem 7.** *If* $f \in L^2(\mathbb{R})$, $g \in L^3(\mathbb{R})$, *and* $h \in L^6(\mathbb{R})$ *then prove that the product* $fgh$ *is in* $L^1(\mathbb{R})$.

*Proof.* Note: $\left\||f|^k\right\|_p = \left(\int |f|^{kp} dx\right)^{1/p} = \left(\int |f|^{kp} dx\right)^{\frac{1}{kp}p} = \|f\|_{kp}^p$.

Then it follows that

$$\|fgh\|_1 \leq \|f\|_2 \|gh\|_2 \leq \|f\|_2 \||g|^2|h|^2\|_1^{1/2} \leq \|f\|_2\big(\||g|^2\|_{p=3/2}\||h|^2\|_{q=3}\big)^{1/3} \leq \|f\|_2\big(\|g\|_3\|h\|_6\big)^{1/3} < \infty.$$

Where we use $p = 3/2$, $q = 3$ so $\frac{1}{p} + \frac{1}{q} = \frac{2}{3} + \frac{1}{3} = 1$. □

**Problem 8.** *(a) A point* $y$ *in a metric space* $Y$ *is isolated if the set* $\{y\}$ *is both open and closed in* $Y$. *Prove taht* $y \in Y$ *is isolated if and only if the complement* $\{y\}^C$ *is not dense in* $Y$.

*Proof.* $\Rightarrow$) If $y$ is isolated, then $\{y\}$ is open. But $\{y\}^c \cap \{y\} = \emptyset$ so $\{y\}^c$ is not dense.

$\Leftarrow$) Trivially, $\{y\}$ is closed since we're in a metric space. Suppose $\{y\}^c$ is not dense in $Y$. Then there exists an open $U \neq \emptyset$ such that $U \cap \{y\}^c = \emptyset$ (since $A$ is dense in $Y \Leftrightarrow$ for all open $U \neq \emptyset$, $U \cap A \neq \emptyset$).

But if $U \cap \{y\}^c = \emptyset$ then $U \subseteq \{y\}^{cc} = \{y\}$ so $U = \{y\}$ is open. □

(b) *Let $X$ be a countable nonempty complete metric space. Prove that the set of isolated points is dense in $X$.*

*Proof.* Let $Y \subseteq X$ be the set of isolated points. Let $X \backslash Y = \{z_j\}_{j=1}^{\infty}$ (or $\{z_j\}_{j=1}^{n}$).

Since the singleton $\{z_k\}$ is not an isolated point, by (a) we know $\{z_k\}^c$ is dense in $X$, so $\overline{\{z_k\}^c} = X$. So each $\{z_k\}^c$ is open and dense in $X$.

By Baire-Category, $Y = \cap_{j=1}^{\infty}\{z_j\}^c$ (or $\cap_{j=1}^{n}\{z_j\}^c$) is also dense in $Y$. □

**Problem 9.** *Suppose that $f \in L^p(\mathbb{R})$ for all $p \in (1,2)$ and that $\sup_{p\in(1,2)} \|f\|_p < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that*

$$\lim_{p\to 2-} \|f\|_p = \|f\|_2.$$

*Proof.* Let $A = \{x \mid |f(x)| \geq 1\}$, $B = \{x \mid |f(x)| < 1\}$. Then by Monotone Convergence Theorem, $\int_A |f|^p dx \nearrow \int_A |f|^2 dx$.

Let $p_n \uparrow 2$. WLOG assume $p_1 = 3/2$. We know on $B$, $|f|^p \leq |f|^{3/2} \in L^1(B)$. By Dominated Convergence Theorem, $\int_B |f|^p dx \to \int_B |f|^2 dx$ which implies $\int_{\mathbb{R}} |f|^p dx \to \int_{\mathbb{R}} |f|^2 dx$.

Therefore, $\|f\|_p^p \to \|f\|_2^2$ so $\|f\|_p^{p/2} \to \|f\|_2$ as $p \to 2$.

Also, since $M = \sup_{p\in(1,2)} \|f\|_p < \infty$, then $\|f\|_p^{p/2-1} \leq M^{p/2-1}$ for all $p \in (1,2)$.

Then $M^{p/2-1} \to 1$ as $p \to 2$ which implies $\|f\|_p^{p/2} - \|f\|_p \to 0$ as $p \to 2$.

So then, $\|f\|_p \to \|f\|_2$ as $p \to 2$ and $\|f\|_2 < \infty$ since $M < \infty$. □

**Problem 10.** *Let $(X, \|\cdot\|)$ be a normed vector space with a subspace $Y$ and let $\|\cdot\|_1$ be another norm on $Y$ that satisfies*

$$\frac{1}{K}\|y\|_1 \leq \|y\| \leq K\|y\|_1, \quad y \in Y,$$

*where $K > 1$ is a fixed constant. Define $S$ to be the set of linear functionals $\phi : X \to \mathbb{R}$ satisfying*

(i) $|\phi(y)| \leq \|y\|_1$, $y \in Y$,

(ii) $|\phi(x)| \leq K\|x\|$, $x \in X$.

*Prove the following statements:*

(a) $\|x\|_2 := \sup\{|\phi(x)| \mid \phi \in S\}$ *defines a norm on $X$.*

*Proof.* See August 2015 □

(b) *For $y \in Y$, $\|y\|_1 = \|y\|_2$.*

*Proof.* See August 2015 □

*(c) The norms $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent on $X$.*

*Proof.* See August 2015 □

# 11 August 2014

**Problem 1.** *For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be continuous, and assume that for every $x \in [0,1]$ the sequence $(f_n(x))$ is decreasing. Suppose that $f_n$ converges pointwise to a continuous function $f$. Show that this convergence is uniform.*

*Proof.* WLOG: by replacing $f_n$ by $f_n(x) - f(x)$, these are still continuous and decreasing pointwise. So we want to prove $f_n \rightrightarrows 0$.

This is precisely Dini's Theorem (aka freebie question).

Fix $\epsilon > 0$ and let $U_n = f_n^{-1}((-1,\epsilon)) = \{x \in X \mid g_n(x) < \epsilon\}$ which is open. Then for all $x$, $f_n(x) \searrow 0$ so there exists $N$ such that for all $n \geq N$, $|f_n(x)| < \epsilon$ which implies $x \in U_n$.

So $[0,1] = \cup_n U_n$. By compactness of $[0,1]$, there exists a finite subcover $U_{n_1}, U_{n_2}, \ldots, U_{n_k}$ for $n_1 < n_2 < \ldots < n_k$ but since $U_n \subseteq U_{n+1}$ then $U_{n_1} \subseteq U_{n_2} \subseteq \ldots \subseteq U_{n_k}$.

Therefore, $[0,1] \subseteq U_{n_k} =: U_N$ so for all $x \in [0,1]$, then $x \in f_N^{-1}((-1,\epsilon)) \Leftrightarrow |f_N(x)| < \epsilon$.

Decreasing $f_n$ implies that for all $n \geq N$, $|f_n(x)| < \epsilon$ for all $x \in [0,1]$. □

**Problem 2.** *Let $f \in L^1(0,\infty)$. For $x > 0$, define*

$$g(x) = \int_0^\infty f(t)e^{-tx}dt.$$

*Prove that $g(x)$ is differentiable for $x > 0$ with derivative*

$$g'(x) = \int_0^\infty -tf(t)e^{-tx}dt.$$

*Proof.* Since

$$\int_0^\infty \int_0^x |tf(t)e^{-ty}|dydt = \int_0^\infty t|f(t)| \left(\int_0^x e^{-ty}dy\right) dt = \int_0^\infty \underbrace{-e^{-tx}}_{\leq 1} |f(t)|dt + \int_0^\infty |f(t)|dt \leq 2\int_0^\infty |f(t)|dt < \infty.$$

By Fubini, $h(x) = \int_0^\infty \int_0^x -tf(t)e^{-ty}dtdy = \int_0^\infty f(t)e^{-tx}dt + c$.

So $h(x) = g(x) + c$.

From the definition of $h$, we know $h'(x) = g'(x)$ and thus $g(x)$ is differentiable. And $h$ is differentiable since it's absolutely continuous. □

**Problem 3.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function such that*

$$\int_a^b f(x)dx = 0 \text{ for every } a < b.$$

*Show that $f(x) = 0$ for almost every $x \in \mathbb{R}$.*

*Proof.* See question 1 from January 2015. ☐

**Problem 4.** *Let $f$ be Lebesgue measurable on $[0,1]$ with $f(x) > 0$ a.e. Suppose $(E_k)$ is a sequence of measurable sets in $[0,1]$ with the property that $\int_{E_k} f(x)dx \to 0$ as $k \to \infty$.*

*Prove that $m(E_k) \to 0$ as $k \to \infty$.*

*Proof.* Let $F_m = \{x \mid f(x) \geq 1/m\}$ so $F_n \subseteq F_{n+1}$.

Since $f(x) > 0$ almost everywhere, then

$$m\left(\cup_{n=1}^{\infty} F_n\right) = \lim_n m(F_n) = 1.$$

Fix $\epsilon > 0$, so there exists $N$ such that $m(F_n^c) < \epsilon/2$ for $n \geq N$. Now,

$$\frac{1}{N} m(E_k \cap F_N) \leq \int_{E_k \cap F_N} f(x)dx \leq \int_{E_k} f(x)dx \to 0 \quad \text{as } k \to \infty.$$

So there exists some $K$ such that $m(E_k \cap F_N) < \epsilon/2$ for all $k \geq K$. Thus,

$$m(E_k) = m(E_K \cap F_N) + m(E_k \cap F_N^c) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \forall k \geq K.$$

☐

**Problem 5.** *Let $(f_n)$ be a sequence of continuous functions on $[0,1]$ such that for each $x \in [0,1]$ there is an $N = N_x$ so that*

$$f_n(x) \geq 0 \text{ for all } n \geq N_x.$$

*Show that there is an open nonempty set $U \subset [0,1]$ and an $N \in \mathbb{N}$, so that $f_n(x) \geq 0$ for all $n \geq N$ and all $x \in U$.*

*Proof.* Let

$$E_n := \{x \mid f_m(x) \geq 0 \ \forall m \geq n\} = \bigcap_{n=m}^{\infty} \{x \mid f_n(x) \geq 0\}$$

so $E_n$ is closed and $E_n \supseteq E_{n+1}$. For all $x \in [0,1]$ there exists $N = N_x$ such taht $f_m(x) \geq 0$ for all $m \geq N$. Thus, $x \in E_N$.

Then, $[0,1] = \cup_{n=1}^{\infty} E_n$. Since $[0,1]$ is compact, by Baire-Category we know there exists $N$ such that $\overline{E_N}^{\circ} \neq \emptyset$ (i.e. $E_N^{\circ} \neq \emptyset$).

Let $U = E_N^{\circ}$ be open, non-empty so for all $x \in U$, $f_n(x) \geq 0$ for all $n \geq N$. □

**Problem 6.** *(a) Define the $w^*$-topology on the dual $X^*$ of a Banach space $X$.*

*Proof.* See wikipedia! □

*(b) Let $X$ be an infinite dimensional Banach space. What is the $w^*$-closure of*

$$S_{X^*} = \{x^* \in X^* \mid \|x^*\| = 1\}?$$

*(as usual, prove your answer.)*

*Proof.* Claim: $\overline{S_{X^*}}^{w^*} = B_{X^*}$.

We know for any $x_1, x_2, \ldots, x_n \in X$, there exists some $x_0^* \neq 0$ such that $x_0^*(x_i) = 0$. Indeed, if this were not true then otherwise, $x_0^*(x_i) \neq 0$ for some $i$, let $\varphi : X^* \to \mathbb{R}^n$ be $\varphi(x^*) = (x^*(x_1), \ldots, x^*(x_n))$ then $\varphi$ is injective so $\dim(X^*) \leq \dim(\mathbb{R}^n) = n$. Contradiction, so true.

Now for any $x^* \in B_{X^*}$, consider it's neighborhood (open under the $w^*$-neighborhood)

$$V = \cap_{i=1}^{n} \{y^* \in X^* \mid |\hat{x_i}(x^* - y^*)| = |x^*(x_i) - y^*(x_i)| < \epsilon\}$$

for each $\{x_i\}_{i=1}^n$ choose such an $x_0^* \neq 0$ from the claim.

Consider the line $\{x^* + tx_0^* \mid t \in \mathbb{R}\}$ in $X^*$.

Since for any $\hat{x_i}$,

$$\hat{x_i}(x^* + tx_0^* - x^*) = t\hat{x_i}(x_0^*) = tx_0^*(x_i) = 0 < \epsilon.$$

Then $\{x^* + tx_0^* \mid t \in \mathbb{R}\} \subseteq V$. Since $\|x^* + tx_0^*\|$ is continuous about $t$, then we can find $t_0$ such that $\|x^* + t_0 x_0^*\| = 1 \Rightarrow V \cap S_{X^*} \neq \emptyset$.

Since any neighborhood of $x^*$ contains a neighborhood of the form $V$ as above (i.e. these $V$'s are a neighborhood basis) then $B_{X^*} \subseteq \overline{S_{X^*}}^{w^*}$.

On the other hand, for any $x_0^* \in B_{X^*}$, by Hahn-Banach separation Theorem, we know there exists $x \in X$ and $c \in \mathbb{R}$ such that $x^*(x) < c < x_0^*(x)$ for all $x^* \in B_{X^*}$.

Then for all $\{x_n^*\} \subseteq B_{X^*}$, $x_n^*(x) \leq c < x_0^*(x)$. Therefore, $x_0^*$ isn't an accumulation point of $B_{X^*}$ which implies $\overline{B_{X^*}}^{w^*} = B_{X^*}$. Thus, $\overline{S_{X^*}}^{w^*} \subseteq \overline{B_{X^*}}^{w^*} = B_{X^*}$ so $B_{X^*} = \overline{S_{X^*}}^{w^*}$. □

**Problem 7.** *(a) State the Riesz Representation Theorem for the dual $L_p^*(\mu)$ of $L_p(\mu)$, $1 \leq p < \infty$.*

*Proof.* See Wikipedia! □

*(b) Let $\mu$ be a finite measure on the measurable space $(\Omega, \Sigma)$. Prove the following part of the above theorem:*

*If $F \in L_p^*(\mu)$, then there exists an $h \in L_1(\mu)$ so that*

$$F(\chi_A) = \int_A h d\mu \text{ for all } A \in \Sigma.$$

*Proof.* Let $\nu(A) = F(\chi_A)$. The goal is to show $\nu$ is a $\sigma$-finite signed measure.

(a) $\nu(\emptyset) = F(\chi_\emptyset) = F(0) = 0$

(b) Let $\{E_i\}$ be disjoint, let $E = \cup_{i=1}^\infty E_i$. Then

$$\begin{aligned} \nu(E) - \sum_{i=1}^n \nu(E_i) &= F(\chi_E) - F\left(\sum_{i=1}^n \chi_i\right) \\ &\leq \|F\|_{L_p^*} \left\| \chi_E - \sum_{i=1}^n \chi_i \right\|_p \\ &\leq \|F\|_{L_p^*} \left\| \sum_{i=n+1}^\infty \chi_i \right\|_p \\ &= \|F\|_{L_p^*} \mu\left(\cup_{i=n+1}^\infty E_i\right)^{1/p} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Therefore, $\nu(E) = \sum_{i=1}^\infty \nu(E_i)$.

When $\mu(A) = 0$, then

$$\nu(A) = F(\chi_A) \leq \|F\|_{L_p^*} \|\chi_A\|_p = \|F\|_{L_p^*} \mu(A)^{1/p} = 0.$$

So $\nu \ll \mu$.

Then from the Radon-Nikodyn Theorem, there exists some $h \in L^1(\mu)$ such that $\nu(A) = \int_A h d\mu$. So

$$F(\chi_A) = \nu(A) = \int_A h d\mu.$$

□

**Problem 8.** *Assume that $(x_n)$ is a weakly converging sequence in a Hilbert space $\mathcal{H}$. Show that there is a subsequence $(y_n)$ of $(x_n)$ so that*

$$\frac{1}{n} \sum_{j=1}^n y_j$$

*converges in norm.*

*Proof.* WLOG $x_n \to 0$ weakly ($\langle x_n, y\rangle \to \langle 0, y\rangle$ for all $y \in \mathcal{H}$) and we know $\|x_n\|$ is bounded, $\sup_n \|x_n\| \le C$. For $n > m$,

$$\begin{aligned}\left\|\frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\right\|^2 &= \left\langle \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j, \frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\right\rangle \\ &= \left\langle \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j, \left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle \\ &\le \left(\frac{1}{m}-\frac{1}{n}\right)^2\left\|\sum_{j=1}^m y_j\right\|^2 + 2\left|\left\langle\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j, \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle\right| + \left(\frac{1}{n}\right)^2\left\|\sum_{j=m+1}^n y_j\right\|^2.\end{aligned}$$

Now by induction, we can choose $y_j$ such that $|\langle y_j, \sum_{n=1}^m y_n\rangle| < 2^{-j}$ for all $m \le j-1$. Pick $y_1$ randomly.

Since $\langle x_n, y\rangle \to \langle 0, y\rangle$ for all $y \in \mathcal{H}$, then we can find $y_2$ such that $\langle y_2, y_1\rangle < 2^{-2}$.

Similarly, find $y_3$ such that $\langle y_3, y_1 + y_2\rangle < 2^{-3}$ and $\langle y_3, y_1\rangle < 2^{-3}$, etc. Then

$$\left\|\sum_{j=1}^m y_j\right\|^2 = \left\langle \sum_{j=1}^m y_j, \sum_{j=1}^m y_j\right\rangle = \langle y_m, y_m\rangle + \langle y_m, \sum_{j=1}^{m-1} y_j\rangle + \ldots + \langle y_1, y_1\rangle \le \sum_{j=1}^m \|y_j\|^2 + \sum_{j=1}^m 2^{-j} < \sum_{j=1}^m \|y_j\|^2 + 2.$$

Therefore,

$$\left(\frac{1}{m}-\frac{1}{n}\right)^2\left\|\sum_{j=1}^m y_j\right\|^2 \le \frac{1}{m^2}\left(\sum_{j=1}^m \|y_j\|^2 + 2\right) < \frac{1}{m^2}(mc+2) \to 0 \text{ as } m \to \infty.$$

Similar argument holds for $\left(\frac{1}{n}\right)^2\left\|\sum_{j=m+1}^n y_j\right\|^2$.

Finally,

$$\begin{aligned}\left|\left\langle\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=1}^m y_j, \frac{1}{n}\sum_{j=m+1}^n y_j\right\rangle\right| &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=m+1}^n \left\langle y_j, \sum_{k=1}^m y_k\right\rangle \\ &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right)\sum_{j=m+1}^n 2^{-(m+1)} \\ &\le \frac{1}{n}\left(\frac{1}{m}-\frac{1}{n}\right) \to 0\end{aligned}$$

So then $\left\|\frac{1}{m}\sum_{j=1}^m y_j - \frac{1}{n}\sum_{j=1}^n y_j\right\|^2 \to 0$ as $n, m \to \infty$ so it's Cauchy and therefore converges. □

**Problem 9.** *Show that a linear functional $\phi$ on a Banach space $X$ is continuous if and only if $\{x \in X \mid \phi(2x) = 3\}$ is norm closed.*

*Proof.* $\Rightarrow$) $A = \{x \mid \phi(2x) = 3)\} = \{x \mid 2x \in \phi^{-1}(\{3\})\}$. Let $\psi(x) = \phi(2x)$ so $A = \psi^{-1}(\phi^{-1}(\{3\}))$.

$\Leftarrow$) We want to show $\ker(\phi)$ is closed. Note that $\{x \in X \mid \phi(2x) = 3\} = \{x \in X \mid \phi(x) = 3/2\}$.

Pick some $a \in X$ such that $\phi(a) = 3/2$. Then clearly

$$a + \ker(\phi) \subseteq \{x \in X \mid \phi(x) = 3/2\}$$

and if $\phi(x) = 3/2$ then $\phi(x - a) = 0$ so $x = a + (x - a) \in a + \ker(\phi)$.

Thus, $a + \ker(\phi) = \{x \in X \mid \phi(2x) = 3\}$. Therefore $\ker(\phi) = \{x \in X \mid \phi(2x) = 3\} - a$ which is closed. Then

$$\begin{aligned} \phi' : X/\ker(\phi) &\to \mathbb{R} \\ x + \ker(\phi) &\mapsto \phi(x) \end{aligned}$$

is an isomorphism. Let $\pi : X \to X/\ker\phi$ which is also continuous, so $\phi = \phi' \circ \pi$ is continuous.

□

**Problem 10.** *Let $C^1[0,1]$ be the space of functions $f \in C[0,1]$ such taht $f'$ exists and is continuous in $[0,1]$. The space $C^1[0,1]$ is given the supremum norm. Define $T : C^1[0,1] \to C[0,1]$ by $Tf = f'$ for $f \in C^1[0,1]$. Show that $T$ has a closed graph and that $T$ is not bounded. Decide if $C^1[0,1]$ (together with the supremum norm) is a Banach space or not. (Explain your answer).*

*Proof.* Let $f_n \to f$ and $Tf_n \to f_n' \to g$ in $\|\cdot\|_\infty$.

$$f_n(x) = \int_0^x f_n'(t)dt + f_n(0) \qquad f(x) = \int_0^x f'(t)dt + f(0)$$

Since $f_n \to f$ then $f_n(0) \to f(0)$. Let $G = \int_0^x g(t)dt + f(0)$. Then

$$\|f - G\| \leq \|f - f_n\| + \|f_n - G\| \leq \|f - f_n\| + \int_0^x \|f_n - g\|_\infty \leq \|f - f_n\| + \|f_n - g\|_\infty \to 0.$$

So $f' = g$ meaning $T$ has a closed graph.

To see $T$ is not bounded, consider $f_n = x^n$ so $\|f_n\|_\infty = 1$ but $\|Tf_n\| = \|nx^{n-1}\|_\infty = n \to \infty$.

Thus, by the closed graph theorem, $C^1[0,1]$ is not a Banach space. □

## 12 January 2014

**Problem 1.** *Let $(X, \mathcal{M}, \mu)$ be a non atomic masure space with $\mu(X) > 0$. Show that there is a measurable $f : X \to [0, \infty)$, for which*

$$\int f(x) d\mu(x) = \infty.$$

*Proof.* Take $X = E_1 \supseteq E_2 \supseteq E_3 \supseteq \dots$ such that $\mu(E_1) > \mu(E_2) > \dots > 0$. Define

$$f(x) = \begin{cases} \mu(E_n \backslash E_{n+1})^{-1} & \text{if } x \in E_n \backslash E_{n+1} \\ 0 & \text{if } x \in \cap_{n=1}^{\infty} E_n \end{cases}$$

Then $\int f(x)dx = \sum_{n=1}^{\infty} 1 = \infty$. □

**Problem 2.** *Assume that $\mu$ is a finite measure on $\mathbb{R}^n$. Prove that there is a closed set $A \subset \mathbb{R}^n$ with the property that for each closed $B \subsetneq A$ it follows that $\mu(A \backslash B) \neq 0$.*

*Proof.* Since $\mathbb{R}^n$ is a locally compact Hausdorff space and $\mu$ is finite, $\mu$ is Radon (and regular). If $\mu(\mathbb{R}^n) = a$ (so $V \neq \mathbb{R}^n$) then we can set

$$M_n := \left\{ U \mid U \text{ is open and } \mu(U) < \frac{a}{n} \right\}$$

and $V := \cup_{n=1}^{\infty} \{U \mid U \in M_n\}$ so $V$ is open. Let $A = V^c$ is closed.

For any $B \subsetneq A$, $A \backslash A \cap B^c$. Assume $\mu(A \backslash B) = 0$ then

$$\mu(A \backslash B) = \inf\{\mu(U) \mid A \backslash B \subseteq U, U \text{ open}\} = 0.$$

Then there exists $U \subseteq V$ such that $A \backslash B \subseteq U$. Then $A \backslash B \cap U \subseteq A \cap U \subseteq A \cap V = \emptyset$, so $A \backslash B = \emptyset$. Contradiction! □

**Problem 3.** *For a nonnegative function $f \in L_1([0, 1])$, prove that*

$$\lim_{n \to \infty} \int_0^1 \sqrt[n]{f(x)} dx = m(\{x \mid f(x) > 0\}).$$

*Proof.* Let

$$E_1 = \{x \mid f(x) \geq 1\}$$
$$E_2 = \{x \mid 0 < f(x) < 1\}$$
$$E_3 = \{x \mid f(x) = 0\}$$

Then

$$\int_0^1 f(x)^{1/n}dx = \int_{E_1} f(x)^{1/n}dx + \int_{E_2} f(x)^{1/n}dx + \int_{E_3} f(x)^{1/n}dx$$

For the first integral on $E_1$, $\lim_n f(x)^{1/n}dx = 1$ and $|f(x)^{1/n}| \leq |f(x)| \in L^1$, so by DCT, $\int_{E_1} f(x)^{1/n}dx = \int_{E_1} dx = m(E_1)$.

For the second integral on $E_2$, $\lim_n f(x)^{1/n} = 1$ and $|f(x)^{1/n}| \leq 1 \in L^1$ so again by DCT, $\int_{E_2} f(x)^{1/n}dx = \int_{E_2} dx = m(E_2)$. Therefore,

$$\int_0^1 f(x)^{1/n}dx = m(E_1) + m(E_2) = m(\{x \mid f(x) > 0\}).$$

□

**Problem 4.** *Let $f$ be Lebesgue integrable on $(0,1)$. For $0 < x < 1$ define*

$$g(x) = \int_x^1 t^{-1}f(t)dt.$$

*Prove that $g$ is Lebesgue integrable on $(0,1)$ and that*

$$\int_0^1 g(x)dx = \int_0^1 f(x)dx.$$

*Proof.* Notice that

$$\int_0^1 \int_0^t t^{-1}f(t)dtdx = \int_0^1 |f(t)|dt < \infty$$

since $f \in L^1(0,1)$ so then by Fubini, we have that

$$\int_0^1 \int_x^1 t^{-1}f(t)dtdx = \int_0^1 \int_0^t t^{-1}f(t)dxdt.$$

So we have

$$\int_0^1 g(x)dx = \int_0^1 \int_x^1 t^{-1}f(t)dtdx = \int_0^1 \int_0^t t^{-1}f(t)dxdt = \int_0^1 f(t)dt.$$

□

**Problem 5.** *Assume that $\nu$ and $\mu$ are two finite measures on a measurable space $(X,\mathcal{M})$. Prove that*

$$\nu << \mu \Leftrightarrow \lim_{n\to\infty} (\nu - n\mu)^+ = 0.$$

*Proof.* $\Leftarrow$) Let $\mu(E) = 0$. Then for all $\epsilon > 0$, there exists some $N$ such that for all $n \geq N$,

$$\epsilon > (\nu - n\mu)^+(E) \geq (\nu - n\mu)(E) = \nu(E) - n\mu(E) \geq \nu(E).$$

Letting $\epsilon$ approach 0, we have that $\nu(E) = 0$ so that $\nu << \mu$. □

**Problem 6.** *Let $(p_n)$ be a sequence of polynomials which converges uniformly on $[0,1]$ to some function $f$, and assume that $f$ is not a polynomial. Prove the $\lim_{n\to\infty} \deg(p_n) = \infty$, where $\deg(p)$ denotes the degree of a polynomial $p$.*

*Proof.* Assume to the contrary and consider the space $\mathcal{P} = \text{span}\{1, x, x^2, \dots, x^m\}$ with $(p_n) \subseteq \mathcal{P}$. Since $\{1, x, \dots, x^m\}$ are basis elements and $\mathcal{P}$ is finite dimensional, then any two norms are equivalent on $\mathcal{P}$ and so if $P = \sum_{k=0}^{\infty} a_k x^k$, we can consider the two norms defined by

$$\|P\|_1 := \sup |a_k| \qquad \|P\|_2 := \sup_{x\in[0,1]} |P(x)|$$

Since $\|p_{n_k} - p_{n_\ell}\|_2 \to 0$, then $\|p_{n_k} - p_{n_\ell}\|_1 \to 0$ so $\{a_{n_k}\}$ is Cauchy. Hense, $P = \sum_{\ell=0}^{m} a_\ell x^\ell$ where $a_\ell = \lim_k a_{n_k}^\ell$ is a polynomial of degree at most $m$ and $p_n$ converges uniformly to $p$. So therefore, $p = f$. Contradiction!

*Alternative proof.* Assume not, so there exists some $M$ such that for all $N \in \mathbb{N}$, there exists some $n \geq N$ such that $\deg(p_n) \leq M$. For $N = 1$, find $n_1$ such that $\deg(p_{n_1}) \leq M$. For $N = n_1$, find $n_2$ such that $\deg(p_{n_2}) \leq M$, etc.

Get a subsequence $\{p_{n_k}\}$ such that $\deg(p_{n_k}) \leq M$.

Since $p_n$ converges to $f$ uniformly on $[0,1]$ then $p_{n_k}$ converges to $f$ uniformly on $[0,1]$.

We write $p_{n_i} := \sum_{j=1}^{m} a_{ij} x^j$, then

$$f = \lim_{i\to\infty} \sum_{j=i}^{m} a_{ij} x^j = \sum_{j=1}^{m} \lim_{i\to\infty} a_{ij} x^j = \sum_{j=1} x^j \qquad \text{where } b_j = \lim_{i\to\infty} a_{ij}.$$

To see $\lim_{i\to\infty} a_{ij}$ exists: let $X = \mathcal{P}p \mid p$ polynomial with degree $\leq M\}$ is a finite dimensional subspace of $\mathcal{C}[0,1]$ hense closed so $f \in X$ so $f$ is a polynomial. □

**Problem 7.** *Let $(f_n)$ be sequence of non zero bounded linear functionals on a Banach space $X$. Show that there is an $x \in X$ so that $f_n(x) \neq 0$, for all $n \in \mathbb{N}$.*

*Proof.* Let $E_n = \{x \mid f_n(x)\}$ which is closed in $X$. Assume the result is not true, so for every $x \in X$, there exists some $n$ such that $f_n(x) = 0$ implies $x \in E_n$, that is, $X = \cup_n E_n$.

Since $X$ is a Banach space, then by Baire Category Theorem, there exists some $n$ such that $\emptyset \neq \overline{E_n^\circ} = E_n^\circ$.

Thus, there exists some $r > 0$, $x \in X$ such that $B(r,x) \subseteq E_n$.Then for all $y \in X$,

$$r\frac{y}{\|y\|} + x \in x + B(r,0) = B(r,x)$$

so then if $f_n(r\frac{y}{\|y\|} + x) = 0$, then $\frac{r}{\|y\|}f_n(y) = -f_n(x) = 0$ so $f_n(y) = 0$ so $f_n = 0$.

Thus, by Baire Category, $\cup C_n \neq X$. Contradiction! □

**Problem 8.** *Assume that $T : \ell_1 \to \ell_2$ is bounded, linear and one-to-one. Prove that $T(\ell_1)$ is not closed in $\ell_2$.*

*Proof.* If $T(\ell^1)$ is closed, then $T(\ell^1)$ is a hilbert space. Since $T : \ell^1 \to T(\ell^1)$ is bijective, by the open mapping theorem. $T$ is open and $T^{-1}$ is bounded so $T$ is an isomorphism. Then $\ell^1 \cong T(\ell^1)$. But $\ell^1$ is not reflexive and $T(\ell^1)$ is reflexive, so contradiction.

*Alternative Proof.* $T(\ell^1)$ is closed, hence complete. So $T : \ell^1 \to T(\ell^1)$ is bijective, so by the open mapping theorem, $T$ is an open map so $T(\ell^1)$ is both open and closed in $\ell^2$. Hence, $T(\ell^1) \cong \ell^2$ so then $\ell^1 \cong \ell^2$. Contradiction! □

**Problem 9.** *For a uniformly bounded sequence $(f_n)$ in $C[0,1]$ (i.e. $\sup_{n\in\mathbb{N}} \sup_{\xi\in[0,1]} |f_n(\xi)| < \infty$) show that $f_n$ converges weakly to 0 $\Leftrightarrow \lim_{n\to\infty} f_n(\xi) = 0$ for all $\xi \in [0,1]$.*

*Is the equivalence true if we do not assume that $(f_n)$ is uniformly bounded, explain?*

*Proof.* This question is the same as 3 from August 2015.

$\Rightarrow$) $\mathcal{C}([0,1])^* = \mathcal{M}[0,1]$ for all $\xi \in [0,1]$, $\delta_\xi \in \mathcal{M}[0,1]$. So $0 = \lim_n \int f_n d\delta_\xi = \lim_n f_n(\xi)$ so then $\lim_n f_n(\xi) = 0$ (note that this does not require uniformly boundedness!)

$\Leftarrow$) Fix $\mu \in \mathcal{M}[0,1]$, we want to show that $\int f_n d\mu \to 0$. Since $|f_n(x)| \leq M$ for all $x$ and all $n$, then by dominated convergence theorem, $\int f_n d\mu \to 0$.

Finally, consider $h_n$ given by connecting $(0,0), (1/n, n), (2/n, 0)$ and $(1,0)$. So $h_n(\xi) \to$ for all $x\imath \in [0,1]$. But by taking Lebesgue measure, $\int h_n(x) d\mu(x) = 1$ so $f_n \nrightarrow 0$ weakly. □

**Problem 10.** *Assume that $f$ is measurable and non negative function on $[0,1]^2$ and that $1 \leq r < p < \infty$. Show that*

$$\left(\int_0^1 \left(\int_0^1 f^r(x,y)dy\right)^{p/r} dx\right)^{1/p} \leq \left(\int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy\right)^{1/r}.$$

*Hint: Let $s = p/r$, let $1 < s' < \infty$ be the conjugate of $s$ and let*

$$F : [0,1] \to \mathbb{R}_0^+, \quad x \mapsto \int_0^1 f^r(x,y)dy.$$

*Then consider for an appropriate function $h \in L_{s'}[0,1]$ the product $hF$.*

*Proof.* Let $F(x) := \int_0^1 f^r(x,y)dy$. Let $h \in L^{s'}[0,1]$ with $\|h\|_{s'} = 1$ and $h \geq 0$. Then by Tonelli (since $Fh \geq 0$), we have

$$\begin{aligned}\int_0^1 F(x)h(x)dx &= \int_0^1 \int_0^1 f^r(x,y)h(x)dydx \\ &= \int_0^1 \int_0^1 f^r(x,y)h(x)dxdy \\ &\leq \int_0^1 \|f^r(\cdot,y)\|_s \|h\|_{s'} dy \\ &= \int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy\end{aligned}$$

So then $\int_0^1 F(x)h(x)dx \leq \int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy$ for all $\|h\|_{s'} = 1$, $h \geq 0$.

Notice that $F \geq 0$ so when $\|\tilde{h}\|_{s'} = 1$, we have

$$\sup_{\|\tilde{h}\|_{s'}=1} \int_0^1 F(x)h(x)dx = \sup_{\|h\|_{s'}=1, h\geq 0} \int_0^1 F(x)h(x)dx$$

Therefore,

$$\sup_{\|h\|_{s'}=1,h\geq 0} \int_0^1 F(x)h(x)dx = \|F\|_s = \left(\int_0^1 \left(\int_0^1 f^r(x,y)dy\right)^{p/r}\right)^{r/p} \leq \int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy$$

So then,

$$\left(\int_0^1 \left(\int_0^1 f^r(x,y)dy\right)^{p/r} dx\right)^{1/p} = \left(\int_0^1 \left(\int_0^1 f^p(x,y)dx\right)^{r/p} dy\right)^{1/r}.$$

□

# 13 August 2013

**Problem 1.** *Let $1 \leq p \leq \infty$ and let $f \in L^p(\mathbb{R})$. For $t \in \mathbb{R}$, let $f_t(x) = f(x-t)$ and consider the mapping $G : \mathbb{R} \to L^p(\mathbb{R})$ given by $G(t) = f_t$. The space $L^p(\mathbb{R})$ is equipped with the usual norm topology.*

*(a) Show that $G$ is continuous if $1 \leq p < \infty$.*

*Proof.* Since $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$, we can choose $g \in C_c^\infty(\mathbb{R})$ such that $\|g - f\|_p < \epsilon$. Then

$$\|f_{t_n} - f_t\|_p \leq \|f_{t_n} - g_{t_n}\|_p + \|g_{t_n} - g_t\|_p + \|g_t - f_t\|_p$$

It's easy to see $\|f_{t_n} - g_{t_n}\|_p$ and $\|g_t - f_t\|_p$ are small since $\|g - f\|_p < \epsilon$. For $\|g_{t_n} - g_t\|_p$, then

$$\|g_{t_n} - g_t\|_p = \left(\int_{\mathbb{R}} |g(x - t_n) - g(x-t)|^p dx\right)^{1/p} \leq \left(\int_A \epsilon^p\right)^{1/p} = \epsilon\mu(A)$$

where for each fixed $t_n \to t$, we can find a bounded set $A \subseteq \mathbb{R}$ such that $\cup_n \operatorname{supp} g_{t_n} \cup \operatorname{supp} g_t \subseteq A$. □

(b) *Find an $f$ for which the mapping $G$ is not continuous when $p = \infty$ (and justify your answer)*

*Proof.* We will take $f = \chi_{[0,1]}$, so

$$\|\chi_{[0,1]}(t_n) - \chi_{[0,1]}(t)\|_\infty = \|\chi_{[t_n,t_n+1)} - \chi_{[t,t+1)}\|_\infty = 1 \qquad \forall n$$

although $t_n \to t$, we have $\|\cdot\|_\infty \nrightarrow 0$. □

(c) *Let $1 \leq p, q \leq \infty$ be conjugate exponents (i.e. satisfying $\frac{1}{p} + \frac{1}{q} = 1$). Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ and show that their convolution $h = f * g$ is continuous. Recall*

$$h(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx.$$

*Proof.* Letting $g_t(x) = g(t - x)$, then we have (by Hölder)

$$|h(t) - h(t_n)| \leq \int_{\mathbb{R}} |f(x)||g(t-x) - g(t_n - x)|dx \leq \|f\|_p\|g_t - g_{t_n}\|_q$$

This goes to zero when $1 < p, q < \infty$ from part (a).

Also notice that

$$h(t) = \int_{-\infty}^{\infty} f(x)g(t-x)dx = \int_{-\infty}^{\infty} f(t-y)g(y)dy = g * f.$$

So when $p = 1$, $q = \infty$ the same is true. □

**Problem 2.** *(a) For $f \in C_{\mathbb{R}}([0,1])$, show that $f \geq 0$ if and only if $\|\lambda - f\|_u \leq \lambda$ for all $\lambda \geq \|f\|_u$, where $\|\cdot\|_u$ denotes the uniform (supremum) norm.*

*Proof.* $\Rightarrow$) Assume there exists some $\lambda \geq \|f\|_u$ such that $\|\lambda - f\|_\infty > \lambda$. Then there exists some $x \in [0,1]$ such that $|\lambda - f(x)| = \lambda - f(x) > \lambda$ so then $f(x) < 0$. Contradiction!

$\Leftarrow$) If there exists some $x$ such that $f(x) < 0$, then if $\lambda \geq \|f\|_\infty$ so $\lambda > 0$. Then $\|\lambda - f\|_\infty \geq \lambda - f(x) > \lambda$. Contradiction! □

(b) *Suppose $E \subseteq C_{\mathbb{R}}([0,1])$ is a closed subspace containing the constant function 1. For $\phi \in E^*$, we define $\phi \geq 0$ to mean $\phi(f) \geq 0$ whenever $f \in E$ and $f \geq 0$. Show $\phi \geq 0$ if and only if $\|\phi\| = \phi(1)$.*

*Proof.* $\Rightarrow$) We have

$$\|\phi\| = \sup_{\|f\|_u=1} |\phi(f)| \geq \phi(1).$$

Also for $\|f\|_u = 1$, we have $1 - f \geq 0$ so $\phi(1-f) \geq 0$ implies $\phi(1) \geq \phi(f)$. Moreover, $\phi(1+f) = \phi(1) + \phi(f) \geq 0$ and so $\phi(1) \geq -\phi(f)$ so then $\phi(1) \geq |\phi(f)|$. Therefore, $\phi(1) = \|\phi\|$

$\Leftarrow$) $\phi(1) = \|\phi\| \geq |\phi(f)|$ for all $\|f\|_u \leq 1$. Assume there exists some $f \geq 0$ but $\phi(f) < 0$.

By rescaling we can assume $\|f\|_u < 1$, so then

$$\|\phi\| \geq \phi\left(\frac{1-f}{\|1-f\|}\right) = \frac{1}{\|1-f\|}(\phi(1) - \phi(f)) \geq \phi(1) - \phi(f) > \phi(1) = \|\phi\|$$

which contradicts! □

(c) *If $\phi \in E^*$ and $\phi \geq 0$, show that there is a bounded linear functional $\psi$ on $C_{\mathbb{R}}([0,1])$ so that $\psi \geq 0$ and the restriction of $\psi$ to $E$ is $\phi$.*

*Proof.* By Hahn-Banach, there exists some $\psi$ which is an extension of $\phi$ such that $\|\psi\| = \|\phi\| = \phi(1) = \psi(1)$. So $\psi \geq 0$ follows from (b). □

**Problem 3.** (a) *Let $\mu$ and $\lambda$ be mutually singular complex measures defined on the same measurable space $(X, \mathcal{M})$ and let $\nu = \mu + \lambda$. Show $|\nu| = |\mu| + |\lambda|$.*

*Proof.* Let $X = E \sqcup F$ be a disjoint union such that $\lambda(F) = mu(E) = 0$. Let $P_2 \sqcup N_2 = E$ be the Hahn-decomposition for $\lambda$. Let $P_3 \sqcup N_3 = F$ be the Hahn-decomposition for $\mu$.

Then $P_1 = P_2 \sqcup P_3$, $N_1 = N_2 \sqcup N_3$ will be the Hahn-decomposition for $\nu = \lambda + \mu$. Then

$$\nu^+(A) = \nu(A \cap P_2) + \nu(A \cap P_3) = \lambda(A \cap P_2) + \mu(A \cap P_3)$$
$$\nu^-(A) = \nu(A \cap N_2) + \nu(A \cap N_3) = \lambda(A \cap N_2) + \mu(A \cap N_3)$$

So then

$$|\nu|(A) = \lambda(A \cap P_2) + \mu(A \cap P_3) + \lambda(A \cap N_2) + \mu(A \cap N_3) = |\mu|(A) + |\lambda|(A)$$

Therefore, $|\nu| = |\mu| + |\lambda|$. □

(b) *Construct a nonzero, atomless Borel measure on $[0,1]$ that is mutually singular with respect to Lebesgue measure.*

*Proof.* here. Maybe Cantor-Lebesgue? □

**Problem 4.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions on $[0,1]$ and suppose that for all $x \in [0,1]$, $f_n(x)$ is eventually nonnegative. Show that there is an open interval $I \subseteq [0,1]$ such that for all n large enough, $f_n$ is nonnegative everywhere on $I$.*

*Proof.* Let $U_N = \cap_{n=N}^{\infty} f_n^{-1}[0,\infty) = \{x \mid f_n(x) \geq 0 \; \forall n \geq N\}$. This is closed. For every $x \in [0,1]$, $f_n(x)$ is eventually non-negative so $[0,1] = \cup_N U_N$.

By Baire-Category, there exists some $N$ such that $\emptyset \neq \overline{U_N}^{\circ} = U_N^{\circ}$. So there exists some open $I \subseteq U_N^{\circ} \subseteq [0,1]$ and then for all $n \geq N$, $f_n \geq 0$ on $I$. □

**Problem 5.** *Let $\mu$ be a nonatomic signed measure on a measure space $(X, \Omega)$, with $\mu(X) = 1$. Show that there is a measureable subset $E \subset X$ with $\mu(E) = 1/2$.*

*Proof.* First notice that for every $\epsilon > 0$, there exists some $E \subseteq X$ with $\mu(E) < \epsilon$. This is because we can recursively divide our set into two non-trivial sets and chose the smaller one.

Therefore, for all $n$, we can find a set $E_n$ such taht $0 < \mu(E_n) < 2^{-n}$. Let $\mathcal{S} = \{E \subseteq X \mid \mu(E) \leq \frac{1}{2}\}$ ordered by inclusion.

Zorn's Lemma implies that there exists a maximal element $E$. If $\mu(E) < \frac{1}{2}$ then we can find some $F \subseteq E^c$ with $0 < \mu(F) < \frac{1}{2} - \mu(E)$ but then $\mu(F \cup E) \leq \mu(F) + \mu(E) \leq \frac{1}{2}$ which contradicts maximality. □

**Problem 6.** *Compute*

$$\lim_{n\to\infty} \int_0^{\infty} \frac{n \sin(x/n)}{x(1+x^2)} dx$$

*and justify your computation.*

*Proof.* Let $f_n(x) = \frac{n\sin(x/n)}{x(1+x^2)}$. Recall that $\lim_{t\to 0} \frac{\sin t}{t} = 1$, so $\lim_n \frac{n\sin(x/n)}{x(1+x^2)} = \frac{1}{1+x^2}$ and since $|\sin(x/n)| \leq x/n$ for $x, n$ positive,

$$\left| \frac{n \sin(x/n)}{x(1+x^2)} \right| \leq \left| \frac{n}{x} \frac{x}{n} \frac{1}{1+x^2} \right| = \frac{1}{1+x^2} \in L^1[0,\infty).$$

Then by DCT,

$$\lim_n \int_0^{\infty} \frac{n\sin(x/n)}{x(1+x^2)} = \int_0^{\infty} \lim_n \frac{n\sin(x/n)}{x(1+x^2)} = \arctan|_0^{\infty} = \frac{\pi}{4}.$$

□

**Problem 7.** *Prove or disprove: for every real-valued continuous function $f$ on $[0,1]$ such that $f(0) = 0$ and every $\epsilon > 0$, there is a real polynomial $p$ having only odd powers of $x$, i.e. $p$ is of the form*

$$p(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots + a_{2n+1} x^{2n+1},$$

*such that* $\sup_{x\in[0,1]} |f(x) - p(x)| < \epsilon$.

*Proof.* Let

$$\mathcal{A} = \{ \text{ polynomial with even power}\}$$

so $\mathcal{A}$ is an algebra that separates points. Stone-Weierstrass implies that $\mathcal{A}$ is dense in $\mathcal{C}[0,1]$. We can let

$$\mathcal{B} = F(x) \mid F(x) = \int_0^x f(t)dt \; 0 \leq x \leq 1, f \in \mathcal{A}\} = \{ \text{ all polynomials with odd powers}\}$$

From problem 10 of August 2015, $\overline{\mathcal{B}} = \mathcal{C}_0[0,1]$. ☐

**Problem 8.** *Let* $f \in L^1_{loc}(\mathbb{R})$.

(a) *What (by definition) are the Hardy-Littlewood maximal function* $Hf$ *and the Lebesgue set* $L_f$ *of* $f$*?*

*Proof.*
$$Hf(x) = \sup_{r>0} \underbrace{\frac{1}{m(B(r,x))} \int_{B(r,x)} |f(y)|dy}_{=:A_r f(x)}.$$

$$L_f = \left\{ x \mid \lim_{r\to 0^+} \frac{\int_{B(r,x)} |f(y) - f(x)|dy}{m(B(r,x))} = 0 \right\}$$

☐

(b) *State the Hardy-Littlewood Maximal Theorem.*

*Proof.* $\|Hf(x)\|_p \leq \|f\|_p$. ☐

(c) *In each case, either construct concretely an example of* $f$ *with the required property, or explain why no such example exists (you may use theorems from Folland about the Lebesgue set, if you state them).*

(i) $L_f = \mathbb{R}$

(ii) *the complement of* $L_f$ *is uncountable*

(iii) $L_f \subseteq (-\infty, 0] \cup [1, \infty)$.

*Proof.* here ☐

**Problem 9.** *Let* $X$ *be a separable Banach space, let* $\{x_n \mid n \geq 1\}$ *be a countable, dense subset of the unit ball of* $X$ *and let* $B$ *be the closed unit ball in the dual Banach space* $X^*$ *of* $X$*. For* $\phi, \psi \in B$*, let*

$$d(\phi,\psi) = \sum_{n=1}^{\infty} 2^{-n} |\phi(x_n) - \psi(x_n)|.$$

*Show that $d$ is a metric on $B$ whose topology agrees with the weak*-topology of $X^*$ restricted to $B$.*

*Proof.* We first check that $d$ is a metric on $B$:

- $d(\phi,\psi) \geq 0$ clear
  If $d(\phi,\psi) = 0$ then $\phi = \psi$ on $\{x_n\}$ so $\phi = \psi$ by continuity / density
- triangle inequality follows as well

the weak*-topology is $\{z^* \in X^* \mid |(x^* - z^*)(x)| < \epsilon\}$ for fixed $x^* \in X^*, x \in X, \epsilon > 0$.

For fixed $x^* \in X^*$ consider the $\epsilon$-ball in the metric $d$ which is

$$\psi \in X^* \mid d(x^*,\psi) < \epsilon\} = \{\psi \in X^* \mid \sum_{n=1}^{\infty} 2^{-n} |x^*(x_n) - \psi(x_n)| < \epsilon\}$$

We want to show that $|(x * -\psi)(x)| < \epsilon'$ for any $x \in B_X$ and some $\epsilon' > 0$. Since $x_n$ is dense in $B_X$ then there exists a $x_{n_k} \to x$. Then

$$|(x^* - \psi)(x)| \leq |(x^* - \psi)(x - x_{n_k})| + |(x^* - \psi)(x_{n_k})| < \epsilon'$$

On the other hand, if $\psi$ is in a weak* neighborhood of $x*$, we want to show $\sum_{n=1}^{\infty} 2^{-n}|x^*(x_n) - \psi(x_n)| < \epsilon$. Let $|(\psi - x^*)(x_n)| < \epsilon'$ for all $n$, then

$$\sum_{n=1}^{\infty} 2^{-n} |x^*(x_n) - \psi(x_n)| < \sum_{n=1}^{\infty} 2^{-n}\epsilon = \epsilon.$$

*Alternative Proof.* We first check that $d$ is a metric on $B$:

- $d(\phi,\psi) \geq 0$ clear
  If $d(\phi,\psi) = 0$ then $\phi = \psi$ on $\{x_n\}$ so $\phi = \psi$ by continuity / density
- triangle inequality follows as well

To see that the topologies agree:

Consider $B(r,\varphi)$ under the metric. We need to show it contains an open $U$ under the weak*-topology. Say $d(\phi_k,\psi) \to 0$. Then $\sum_{n=1}^{\infty} 2^{-n}|\phi_k(x_n) - \psi(x_n)| \to 0$. So under the weak* topology, we need to show for all $x \in B_X$, $|\phi_k(x) - \psi(x)| \to 0$.

Indeed, this follows by density of $\{x_n\}$. For large $k$, $\|\phi_k\| = \sup_n |\phi_k(x_n)| \leq M$ and $|\phi_k(x_n)| \sim |\psi(x_n)|$.

Then for every $\epsilon > 0$, there exists some $n$ such that $\|x_n - x\| < \epsilon$ so

$$|\phi_k(x) - \psi(x)| \leq |\phi_k(x) - \phi_k(x_n)| + |\phi_k(x_n) - \psi(x_n)| + |\psi(x_n) - \psi(x)|$$

If $|\phi_k(x) - \psi(x)| \to 0$ for all $x$, then for all $\epsilon$, choose $N$ such that $\sum_{n=N}^{\infty} 2^{-n} < \epsilon$, so $d(\phi_k, \psi) = \sum_{n=1}^{\infty} 2^{-n}|\phi_k(x_n) - \psi(x_n)|$. □

**Problem 10.** *Let $T : X \to Y$ be a linear map between Banach spaces that is surjective and satisfies $\|Tx\| \geq \epsilon\|x\|$ for some $\epsilon > 0$ and all $x \in X$. Show that $T$ is bounded.*

*Proof.* Is $\Gamma(T) = \{(x, Tx) \mid x \in X\}$ closed in $X \times Y$?

If $x_n \to x$ and $Tx_n \to y$, we want to show $y = Tx$. $T$ is surjective, so $y = Tx_0$. Then for all $\tilde{\epsilon} > 0$, there exists $N$ such that for all $n \geq N$,

$$\tilde{\epsilon} > \|Tx_n - Tx_0\| \geq \epsilon\|x_n - x_0\|.$$

So $x_n \to x_0$, and $Tx_n \to Tx_0$.

The closed graph theorem implies $T$ is bounded. □

# 14 January 2013

**Problem 1.** *Let $f$ be a Lebesgue integrable, real-valued function on $(0,1)$ and for $x \in (0,1)$ define*

$$g(x) = \int_x^1 t^{-1} f(t)dt.$$

*Show that $g$ is Lebesgue integrable on $(0,1)$ and that $\int_0^1 g(x)dx = \int_0^1 f(x)dx$.*

*Proof.* See January 2014, # 4 □

**Problem 2.** *Let $f_n \in C[0,1]$. Show that $f_n \to 0$ weakly if and only if the sequence $(\|f_n\|)_{n=1}^{\infty}$ is bounded and $f_n$ converges pointwise to 0.*

*Proof.* See August 2015, # 3 □

**Problem 3.** *Let $(X, \mu)$ be a measure space with $0 < \mu(X) \leq 1$ and let $f : X \to \mathbb{R}$ be measurable. State the definition of $\|f\|_p$ for $p \in [1, \infty]$. Show that $\|f\|_p$ is a monotone increasing function of $p \in [1, \infty)$ and that $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.*

*Proof.* See January 2016, # 8 □

**Problem 4.** *(a) Is there a signed Borel measure $\mu$ on $[0,1]$ such that*

$$p'(0) = \int_0^1 p(x)d\mu(x)$$

*for all real polynomials $p$ of degree at most 19?*

*Proof.* We first define the linear functional $I(p) = p'(0)$.

Write $\mathcal{P} = \text{span}\{1, x, x^2, \ldots, x^{19}\}$, which is a finite dimensional space. Thus, all norms are equivalent. We take, in particular, the norms $\|\cdot\|_m = \max_{i=1,\ldots,19} |a_i|$ and $\|\cdot\|_\infty$. Then there must exist some $C$ such that if $\|p\|_\infty = 1$ then $\|p\|_m \leq C$ so $|a_1| \leq C$ which implies that $I$ is bounded.

By Hahn-Banach, there exists some $\tilde{I} \in \mathcal{C}[0,1]^*$ such that $\tilde{I}(p) = I(p)$ for all $p \in \mathcal{P}$. By Riesz, there exists some $\mu$ such that $\tilde{E}(p) = p'(0) = \int_0^1 p(x)d\mu$. □

*(b) Is there a signed Borel measure $\mu$ on $[0,1]$ such that*

$$p'(0) = \int_0^1 p(x)d\mu(x)$$

*for all real polynomials $p$?*

*Proof.* Suppose there did exist such a measure $\mu$ on $[0,1]$. Then since $\mu([0,1]) = \int_0^1 1d\mu = 0$, we have that $|\mu|([0,1]) < \infty$. Therefore, the mapping $T : p \mapsto \int_0^1 p(x)d\mu(x)$ can extend continuously to $C[0,1]$.

Consider $f_n(x)$ defined by $nx$ for $x \in [1/n, 2/n]$, $f_n(x) = 1$ for $x \in [2/n, 1]$ and smooth on the interval $[1/n, 2/n]$ but bounded above by 2 (it's always possible to construct such an $f_n$). Then

$$f_n'(0) = \int_0^1 f_n(x)d\mu \leq \|f_n\|_\infty \|\mu\| \leq \|\mu\| < \infty$$

But $\lim_n |f_n'(0)| = \lim_n = \infty$. Contradiction! □

**Problem 5.** *Let $\mathcal{F}$ be the set of all real-valued functions on $[0,1]$ of the form*

$$f(t) = \frac{1}{\Pi_{j=1}^n (t - c_j)}$$

*for natural numbers $n$ and for real numbers $c_j \notin [0,1]$. Prove or disprove: for all continuous, real-valued functions $g$ and $h$ on $[0,1]$ such that $g(t) < h(t)$ for all $t \in [0,1]$, there is a function $a \in \text{span}\,\mathcal{F}$ such that $g(t) < a(t) < h(t)$ for all $t \in [0,1]$.*

*Proof.* Let $\mathcal{A} = \text{span}\,\mathcal{F}$. It's easy to see this is an algebra since $c_j \notin [0,1]$. Also $\frac{1}{t+1}$ separates points, so Stone-Weierstrass theorem implies $\overline{\mathcal{A}} = C[0,1]$.

Let $M = \min_{t\in[0,1]} |h(t) - g(t)|$, so we can choose some $a \in \mathcal{A}$ such that $\left\| a - \frac{h+g}{2} \right\|_\infty < \frac{M}{6}$. Then $\frac{-M}{6} < a - \frac{h+g}{2} < \frac{M}{6}$ and since $h - g \geq M$, then

$$h - a = \frac{h}{2} - a + \frac{h}{2} \geq \frac{h}{2} - a + \frac{g}{2} + \frac{M}{2} = \frac{h+g}{2} - a + \frac{M}{2} > \frac{M}{2} - \frac{M}{6} = \frac{M}{3} > 0$$

$$a - g = a - \frac{g+g}{2} \geq a - \frac{g+h}{2} + \frac{M}{2} > \frac{-M}{6} + \frac{M}{2} = \frac{M}{3} > 0$$

So then $g < a < h$. □

**Problem 6.** *Let $k : [0,1] \times [0,1] \to \mathbb{R}$ be continuous and let $1 < p < \infty$. For $f \in L^p[0,1]$, let $Tf$ be the function on $[0,1]$ defined by*

$$(Tf)(x) = \int_0^1 k(x,y)f(y)dy.$$

*Show that $Tf$ is a continuous function on $[0,1]$ and that the image under $T$ of the unit ball in $L^p[0,1]$ has compact closure in $C[0,1]$.*

*Proof.* Note that

$$|Tf(x) - Tf(y)| \leq int_0^1 |k(x,z) - k(y,z)||f(z)|dz \leq \|k(x,\cdot) - k(y,\cdot)\|_q \|f\|_p \qquad \text{for } q = \frac{p}{p-1}$$

Since $k$ is continuous on $[0,1]^2$, then for every $\epsilon > 0$ there exists some $\delta > 0$ such that if $|x - y| < \delta$, then

$$\|k(x,\cdot) - k(y,\cdot)\|_q^q = \int_0^1 |k(x,z) - k(y,z)|^q dz < \int_0^1 \epsilon^p dz = \epsilon^p.$$

Therefore, $Tf$ is continuous.

Now consider $\mathcal{F} = \{Tf \mid \|f\|_p \leq 1\} \subseteq C[0,1]$. We'll use Arzela-Ascoli:

- equicontinuous

  follows from above

- pointwise bounded

  $$|Tf(x)| \leq \|K(x,\cdot)\|_q \|f\|_p \leq \|K(x,\cdot)\|_q \leq \left(\int_0^1 M^q dz\right)^{1/q} = M$$

  so it's actually uniformly bounded

Therefore, by Arzela-Ascoli, $\overline{\mathcal{F}}$ is compact in $C[0,1]$. □

**Problem 7.** *(a) Define the total variation of a function $f : [0, 1] \to \mathbb{R}$ and absolute continuity of $f$.*

*Proof.* here □

*(b) Suppose $f : [0, 1] \to \mathbb{R}$ is absolutely continuous and defines $g \in C[0, 1]$ by*

$$g(x) = \int_0^1 f(xy)dy.$$

*Show that $g$ is absolutely continuous.*

*Proof.* Since $f$ is absolutely continuous, there exists some $\delta' > 0$ such that $\sum_{i=1}^n |b_i - a_i| < \delta'$ implies $\sum_{i=1}^n |f(b_i) - f(a_i)| < \epsilon$. Fix some $y \in [0, 1]$ so that

$$\sum_{i=1}^n |b_i y - a_i y| \leq \sum_{i=1}^n |b_i - a_i| < \delta'$$

This implies then that $\sum_{i=1}^n |f(b_i y) - f(a_i y)| < \epsilon$. Therefore,

$$\sum_{i=1}^n |g(b_i) - g(a_i)| \leq \int_0^1 \sum_{i=1}^n |f(b_i) - f(a_i)| \leq \int_0^1 \epsilon dx = \epsilon.$$

So $g$ is absolutely continuous. □

**Problem 8.** *(a) State the definition of absolute continuity, $v << \mu$, for positive measures $\mu$ and $\nu$, and state the Radon-Nikodym Theorem, (or the Lebesgue-Radon-NIkodym Theorem, if you prefer.)*

*Proof.* here □

*(b) Suppose that we have $\nu_1 << \mu_1$ and $\nu_2 << \mu_2$ for positive measures $\nu_i$ and $\mu_i$ on measurable spaces $(X_i, \mathcal{M}_i)$ for $i = 1, 2$. Show that we have $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$, and*

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x, y) = \frac{d\nu_1}{d\mu_1}(x)\frac{d\nu_2}{d\mu_2}(y).$$

*Proof.* Assume $E \in \mathcal{M}_1 \otimes \mathcal{M}_2$ and $\mu_1 \times \mu_2(E) = 0$. Define

$$E_x = \{y \in X_2 \mid (x, y) \in E\} \qquad E^y = \{x \in X \mid (x, y) \in E\}$$

Then $E_x \in \mathcal{M}_1$ and $E^y \in \mathcal{M}_2$ for all $x \in X_1, y \in X_2$. Since $\mu_1$ and $\mu_2$ are positive, then $0 = (\mu_1 \times \mu_2)(E) = \int \mu_1(E^y)d\mu_2(y)$ then $\mu_1(E^y) = 0$ $\mu_2$-almsot everywhere and so then $\nu_1(E^y) = 0$ $\mu_2$-almost everywhere.

Thus, $\mu_2(\{y \in X_2 \mid \nu_1(E^y) > 0\}) = 0$ so then $\nu_2(\{y \in X_2 \mid \nu_1(E^y) > 0\}) = 0$. Thus, $\nu_1(E^y) = 0$ for $\nu_2$-almost everywhere and therefore, $(\nu_1 \times \nu_2)(E) = \int \nu_1(E^y)d\nu_2(y) = 0$.

Thus, $\nu_1 \times \nu_2 << \mu_1 \times \mu_2$. By Radon-Nikodym theorem,

$$\nu_1 \times \nu_2(E) = \int_E \frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y) d(\mu_1 \times \mu_2) \qquad \text{for } E \in \mathcal{M}_1 \otimes \mathcal{M}_2$$

Since $\nu_1 << \mu_1$, by Proposition 3.9(a) in Folland,

$$\begin{aligned}(\nu_1 \times \nu_2)(E) &= \int \nu_2(E_x) d\nu_1(x) \\ &= \int \nu_2(E_x) \frac{d\nu_1}{d\mu_1}(x) d\mu_1(x) \\ &= \int \left( \int_{E_x} \frac{d\nu_2}{d\mu_2}(y) d\mu_2(y) \right) \frac{d\nu_1}{d\mu_1}(x) d\mu_1(x) \\ &= \int_E \frac{d\nu_2}{d\mu_2}(y) \frac{d\nu_1}{d\mu_1}(x) d(\mu_1 \times \mu_2)(x,y)\end{aligned}$$

By the uniqueness of Radon-Nikodym derivative, we have

$$\frac{d(\nu_1 \times \nu_2)}{d(\mu_1 \times \mu_2)}(x,y) = \frac{d\nu_1}{d\mu_1}(x) \frac{d\nu_2}{d\mu_2}(y).$$

□

**Problem 9.** *(a) Let $E$ be a nonzero Banach space and show that for every $x \in E$, there is $\phi \in E^*$ such that $\|\phi\| = 1$ and $|\phi(x)| = \|x\|$.*

*Proof.* This is the Hahn-Banach separation Theorem. □

*(b) Let $E$ and $F$ be Banach spaces, let $\pi : E \to F$ be a bounded linear map and let $\pi^* : F^* \to E^*$ be the induced map on dual spaces. Show that $\|\pi^*\| = \|\pi\|$.*

*Proof.* We have $\pi^*(y^*)(x) = y^*(\pi(x))$ for all $y^* \in F^*$ and $x \in E$. Then $\|\pi^*(y^*)(x)\| \leq \|y^*\|\|\pi\|\|x\|$ so then $\|\pi^*\| \leq \|\pi\|$.

On the other hand, by part (a), for each $x \in E$ such that $\|x\| \leq 1$, $\pi(x) \in F$, we can find $y^* \in F^*$ such that $|y^*(\pi(x))| = \|\pi(x)\|$ and $\|y^*\| = 1$. Then

$$\|\pi^*\| \geq \|\pi^*(y^*)\| \geq |\pi^*(y)(x)| = |y^*(\pi(x))| = \|\pi(x)\| \qquad \forall \|x\| \leq 1$$

So $\|\pi^*\| \geq \|\pi\|$. THus, $\|\pi\| = \|\pi^*\|$. □

**Problem 10.** *Let $X$ be a real Banach space and suppose $C$ is a closed subset of $X$ such that*

*(i) $x_1 + x_2 \in C$ for all $x_1, x_2 \in C$,*

*(ii) $\lambda x \in C$ for all $x \in C$ and $\lambda > 0$,*

*(iii) for all $x \in X$ there exists $x_1, x_2 \in C$ such that $x = x_1 - x_2$.*

*Prove that, for some $M > 0$, the unit ball of $X$ is contained in the closure of*

$$\{x_1 - x_2 \mid x_i \in C, \|x_i\| \le M\}.$$

*Deduce that every $x \in X$ can be written $x = x_1 - x_2$, with $x_i \in C$ and $\|x_i\| \le 2M\|x\|$.*

*Proof.* Define

$$C_n = \overline{\{x_1 - x_2 \mid x_i \in C, \|x_i\| \le n\}}$$

By (iii), we know that $X = \cup C_n$. By Baire Category, there exists some $M$ such that $\emptyset \neq \overline{C_M}^\circ = C_M^\circ$. Thus, there exists an open ball $B \subseteq C_M$, $B = B(x_0, 2r)$.

For any $x \in B_X$, $x_0 + rx \in B \subseteq C_M$. From (i), we know that $C_M - C_M \subseteq C_{2M}$ so then $rx = (x_0 + rx) - x_0 \in C_M - C_M \subseteq C_{2M}$. From (ii), we know $x \in C_{2M/r}$, so $B_X \subseteq C_{2M/r}$. Let $M' = \frac{2M}{r}$.

For any $x \in X$, $x \in C_{M'\|x\|}$. So we can find $z_1, y_1 \in C$ such that $\|z_1\|, \|y_1\| \le M\|x\|$ and $\|x - (z_1 - y_1)\| < \frac{1}{2}\|x\|$. Therefore,

$$\frac{2(x - (z_1 - y_1))}{\|x\|} \in C_M \quad \Rightarrow \quad x - (z_1 - y_1) \in C_{M\|x\|/2}$$

So we can find $z_2, y_2 \in C$ such that $\|z_2\|, \|y_2\| \le \frac{M}{2}\|x\|$ and

$$\left\| x - \sum_{i=1}^{2} (z_i - y_i) \right\| < \frac{1}{2^2}\|x\|.$$

Inductively, we can find $\{z_n\}, \{y_n\} \subseteq C$ such that $\|z_k\|, \|y_k\| \le \frac{M}{2^{k-1}}\|x\|$ and

$$\left\| x - \sum_{i=1}^{k} (z_i - y_i) \right\| < \frac{1}{2^k}\|x\|.$$

Then,

$$\sum_{k=1}^{\infty} \|z_k\| \le \sum_{k=1}^{\infty} M\|x\| \frac{1}{2^k} < 2M\|x\| < \infty$$

so $\sum_{k=1}^{\infty} z_k$ converges to some $x_1$ in $C$ and similarly $\sum_{k=1}^{\infty} y_k$ converges to some $x_2$ in $C$ (since $C$ is closed). Moreover,

$$\lim_n \left\| x - \sum_{i=1}^{n} (z_i - y_i) \right\| = \lim_n \left\| x - \left( \sum_{i=1}^{n} z_i - \sum_{i=1}^{n} y_i \right) \right\| = 0.$$

So then $x = \sum_{i=1}^{\infty}(z_i - y_i) = x_1 - x_2$. □

# 15 August 2012

**Problem 1.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that the normed vector space $L^1(X, \mu)$ is complete. You may use any results except the convergence of function series.*

*Proof.* See class notes. □

**Problem 2.** *Fix two measure spaces $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ with $\mu(X), \nu(Y) > 0$. Let $f : X \to \mathbb{C}$, $g : Y \to \mathbb{C}$ be measurable. Suppose $f(x) = g(y)$, $(\mu \times \nu)$-a.e. Show that there is a constant $a \in \mathbb{C}$ such that $f(x) = a$ $\mu$-a.e. and $g(y) = a$ $\nu$-a.e.*

*Proof.* Let $E := \{(x, y) \in X \times Y \mid f(x) = g(y)\}$, so $(\mu \otimes \nu)(E^c) = 0$. Then for every $a \in \mathbb{C}$, by Fubini-Tonelli,

$$0 = (\mu \otimes \nu)\left(\{(x, y) \in X \times Y \mid f(x) = a, g(y) \neq a\}\right) = \mu\left(\{x \in X \mid f(x) = a\}\right)\nu\left(\{y \in Y \mid g(y) \neq a\}\right).$$

Assume $\mu(\{x \in X \mid f(x) = a\}) = 0$ for all $a \in \mathbb{C}$. Then

$$\begin{aligned}
0 &< (\mu \otimes \nu)(X \times Y) \\
&= (\mu \otimes \nu)(E) = \int_{X \times Y} \chi_{\{(x,y)|f(x)=g(y)\}} d\mu(x) d\nu(y) \\
&= \int_Y \left( \int_X \chi_{\{x|f(x)=g(y)\}} d\mu(x) \right) d\nu(y) \\
&= \int_Y 0 d\nu(y) = 0.
\end{aligned}$$

This is a contradiction so there must exist some $a \in \mathbb{C}$ with $\mu(\{x \in X \mid f(x) = a\}) > 0$. Then $\nu(\{y \in Y \mid g(y) \neq 0\}) = 0$ so $g(y) = a$ $\nu$-a.e.

Similarly, we have $(\mu \otimes \nu)(\{(x, y) \mid f(x) \neq a, g(y) = a\}) = 0$. Since $\nu(\{y \in Y \mid g(y) = a\}) = \nu(Y) \neq 0$, then $\mu(\{x \mid f(x) \neq a\}) = 0$ so $f(x) = a$ $\mu$-a.e. □

**Problem 3.** *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a Borel measurable function. Suppose for every ball $B$, $f$ is Lebesgue integrable on $B$ and $\int_B f(x)dx = 0$. What can you deduce about $f$? Justify your answer carefully.*

*Proof.* Since $f \in L^1_{loc}(\mathbb{R}^2)$, by Lebesgue Differentiation Theorem, for a.e. $x_0 \in \mathbb{R}^2$,

$$\lim_{r \to 0} \frac{1}{|B(r, x_0)|} \int_{B(r,x_0)} f(x)dx = f(x_0)$$

This implies $f(x_0) = 0$ so $f = 0$ almost everywhere. □

**Problem 4.** *Let $X$ be a locally compact Hausdorff space. Denote by $C_0(X)$ the space of complex-valued continuous functions on $X$ which vanish at infinity, and by $C_c(X)$ the subset of compactly supported functions. Use an approximate version of the Stone-Weierstrass theorem to prove that $C_c(X)$ is dense in $C_0(X)$.*

*Proof.* For any $f, g \in C_c(X)$, the complex conjugation of $f$ is also in $C_c(X)$.

By complex-LCH-Stone-Weierstrass, we only need to show that $C_c(X)$ separates points.

For every $x \neq y$, we can find open $U, V$ with $x \in U, y \in V$ with $U \cap V = \emptyset$. Since $X$ is LCH, we can require $\overline{U}$ to be compact.

Now $\{x\} \subseteq U \subseteq \overline{U} \subseteq X \backslash V \subseteq X \backslash \{y\}$. Then by Urysohn's Lemma for LCH, we can find a continuous function $f : X \to [0, 1]$ such taht $f|_{\overline{U}} = 1$ and $f(x) = 0$ outside a compact subset of $X \backslash \{y\}$. So $f(x) = 1$, $f(y) = 0$, and $f \in C_c(X)$.

So $C_c(X)$ separates points. Also, there does not exist any $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in C_c(X)$.

Therefore, by Stone-Weierstrass, $\overline{C_c(X)} = C_0(X)$. □

**Problem 5.** *Give an example of each of the following. Justify your answers*

*(a) A nowhere dense subset of $\mathbb{R}$ of positive Lebesgue measure*

*Proof.* Take a fat Cantor set. □

*(b) A closed, convex subset of a Banach space with multiple points of minimal norm.*

*Proof.* Let $X = L^1[0, 1]$, $C = \{f \in X \mid \int_0^1 f(t)dt = 0\}$. It's easy to see that $C$ is closed and convex. The minimum norm of elements in $C$ is 1 because

$$\|f\|_1 = \int_0^1 |f(t)|dt \geq \left| \int_0^1 f(t)dt \right| = 1.$$

But every element of $\{a\chi_{[0,1/2]} + (2 - a)\chi_{[1/2,1]}\}_{0 \leq a \leq 2}$ in $C$ has norm 1. □

**Problem 6.** *Let*

$$S = \left\{ f \in L^\infty(\mathbb{R}) \mid |f(x)| \leq \frac{1}{1 + x^2} \ a.e. \right\}.$$

*Which of the following statements are true? Prove your answers.*

*(a) The closure of $S$ is compact in the norm topology*

*Proof.* NO. Let

$$f_n(x) := \frac{1}{x^2+1}\chi_{[\frac{1}{n+1},\frac{1}{n}]}(x)$$

in $S$. So there are no subsequences of $(f_n)$ which are Cauchy in $L^\infty$ since $\|f_n - f_m\|_\infty \geq 1$ for $n \neq m$. □

(b) *S is closed in the norm topology.*

*Proof.* YES. Suppose $(f_n) \subseteq S$, $f_n \to f$ in $L^\infty$. Then

$$|f(x)| \leq |f_n(x)| + |f_n(x) - f(x)| < \frac{1}{1+x^2} + \|f_n - f\|_\infty < \frac{1}{1+x^2} + \epsilon \quad \text{a.e.}$$

Letting $\epsilon \to 0$, we have $|f(x)| \leq \frac{1}{1+x^2}$ a.e. and $f \in L^\infty$ so $f \in S$. □

(c) *The closure of S is compact in the weak* topology*

*Proof.* YES. The unit ball in $L^\infty(\mathbb{R})$ is weak*-compact by Alaoglu. Since $\frac{1}{1+x^2} \leq 1$ for all $x \in \mathbb{R}$, then $S$ is a subset of the unit ball in $L^\infty$. Therefore, $\overline{S}^{w*}$ is weak* compact. □

**Problem 7.** *Let $T$ be a bounded operator on a Hilbert space $\mathcal{H}$. Prove taht $\|T^*T\| = \|T\|^2$. State the results you are using.*

*Proof.* Clearly, $\|T^*T\| \leq \|T^*\|\|T\| = \|T\|^2$. On the other hand,

$$\|T\|^2 = \sup_{\|x\|=1} |\langle Tx, Tx\rangle| = \sup_{\|x\|=1} |\langle T^*Tx, x\rangle|.$$

Since for $\|x\| = 1$,

$$|\langle T^*Tx, x\rangle| \leq \|T^*Tx\|\|x\| \leq \|T^*T\|\|x\|^2 \leq \|T^*T\|$$

then $\|T\|^2 \leq \|T^*T\|$. □

**Problem 8.** (a) *Let $g$ be an integrable function on $[0,1]$. Does there exist a bounded measurable function $f$ such that $\|f\|_\infty \neq 0$ and $\int_0^1 fg\,dx = \|g\|_1\|f\|_\infty$? Give a construction or a counterexample.*

*Proof.* YES. For any $g \in L^1$, let $f = \text{sgn}(g)$ where $g(x) \neq 0$, and 1 where $g(x) = 0$. Then $\|f\|_\infty = 1$ and

$$\int_0^1 fg = \int_0^1 |g(x)|dx = \|g\|_1 = \|g\|_1\|f\|_\infty.$$

□

(b) *Let $g$ be a bounded measurable function on $[0,1]$. Does there exist an integrable function $f$ such that $\|f\|_1 \neq 0$ and $\int_0^1 fg dx = \|g\|_\infty \|f\|_1$? Give a construction or a counterexample.*

*Proof.* NO. Let $g(x) = x$ on $[0,1]$ so $\|g\|_\infty = 1$, implying $g \in L^\infty[0,1]$. Assume such an $f \in L^1$ exists, so

$$\|f\|_1 = \|f\|_1 \|g\|_\infty = \int_0^1 fg dx = \int_0^1 x f(x) dx$$

and also $\|f\|_1 = \int_0^1 |f| dx$ so then $\int_0^1 f(x) x dx = \int_0^1 |f| dx$. Therefore,

$$\int_0^1 |f(x)| dx = \int_0^1 x f(x) dx \leq \int_0^1 x |f(x)| dx \leq \left(1 - \frac{1}{n}\right) \int_0^{1-1/n} |f(x)| dx + \int_{1-1/n}^1 |f(x)| dx$$

So then

$$\int_0^{1-1/n} |f(x)| dx + \int_{1-1/n}^1 |f(x)| dx \leq \left(1 - \frac{1}{n}\right) \int_0^{1-1/n} |f(x)| dx + \int_{1-1/n}^1 |f(x)| dx$$

Thus, $\int_0^{1-1/n} |f(x)| dx = 0$ for all $n \in \mathbb{N}$. Letting $f_n(x) = \chi_{[0,1-1/n]} |f(x)| \nearrow |f(x)|$ then by monotone convergence theorem, $\int |f(x) dx = \lim_n \int f_n(x) = 0$ so $\|f\|_1 = 0$. □

**Problem 9.** *Let $F : \mathbb{R} \to \mathbb{C}$ be a bounded continuous function, $\mu$ the Lebesgue measure, and $f, g \in L^1(\mu)$. Let*

$$\tilde{f}(x) = \int F(xy) f(y) d\mu(y), \qquad \tilde{g}(x) = \int F(xy) g(y) d\mu(y).$$

*Show that $\tilde{f}$ and $\tilde{g}$ are bounded continuous functions which satisfy*

$$\int f \tilde{g} d\mu = \int \tilde{f} g d\mu.$$

*Proof.* We have $\|\tilde{f}\|_{|} infty \leq \|F\|_\infty \|f\|_1 < \infty$ and $\|\tilde{g}\| \leq \|F\|_\infty \|g\|_1 < \infty$ so $\tilde{f}, \tilde{g} \in L^\infty$. By dominated convergence theorem, we know that $\lim_n \int_{[-n,n]} |f(x) d\mu = \|f\|_1$. So then for every $\epsilon > 0$, there exists some $N$ such that $\int_{\mathbb{R} \setminus [-n,n]} |f(x)| d\mu < \epsilon$. Then

$$\begin{aligned}
|\tilde{f}(x_1) - \tilde{f}(x_2)| &\leq \int |F(x_1 y) - F(x_2 y)| |f(y)| d\mu(y) \\
&= \int_{[-n,n]} |F(x_1 y) - F(x_2 y)| |f(y)| d\mu(y) + \int_{\mathbb{R} \setminus [-n,n]} |F(x_1 y) - F(x_2 y)| |f(y)| d\mu(y) \\
&\leq \sup_{y \in [-n,n]} |F(x_1 y) - F(x_2 y)| \|f\|_1 + 2\|F\|_\infty \epsilon
\end{aligned}$$

Since $F$ is continuous, let $|x_1 - x_2| < \frac{\delta}{n}$ such that $|x_1 y - x_2 y| < \delta$ imples $|F(x_1 y) - F(x_2 y)| < \epsilon$. So $|\tilde{f}(x_1) - \tilde{f}(x_2)| \to 0$ as $|x_1 - x_2| \to 0$.

A similar argument will show that $\tilde{g}$ is continuous. Since $f\tilde{g} \in L^1$, by Fubini,

$$\begin{aligned}\int f\tilde{g}d\mu &= \int\int f(x)F(xy)g(y)d\mu(y)d\mu(x)\\ &= \int g(y)\left(\int f(x)F(xy)d\mu(x)\right)d\mu(y)\\ &= \int g(y)\tilde{f}(y)d\mu(y)\\ &= \int \tilde{f}gd\mu.\end{aligned}$$

□

**Problem 10.** *Let $\mu$, $\{\mu_n \mid n \in \mathbb{N}\}$ be finite Borel measures on $[0,1]$. $\mu_n \to \mu$ vaguely if it converges in the weak* topology on $M[0,1] = (C[0,1])^*$. $\mu_n \to \mu$ in moments if for each $k \in \{0\} \cup \mathbb{N}$,*

$$\int_{[0,1]} x^k d\mu_n(x) \to \int_{[0,1]} x^k d\mu(x).$$

*Show that $\mu_n \to \mu$ vaguely if and only if $\mu_n \to \mu$ in moments.*

*Proof.* $\Rightarrow$) trivial by the definitions

$\Leftarrow$) We want to show that for all $f \in C[0,1]$, $\int f d\mu_n \to \int f d\mu$. By Stone-Weierstrass, we can find $p_n$ to be a sequence of polynomials which converge uniformly to $f$ on $[0,1]$

$$\left|\int f(x)d\mu - \int f(x)d\mu_n\right| \leq \left|\int fd\mu - \int p_m d\mu\right| + \left|\int p_m d\mu - \int p_m d\mu_n\right| + \left|\int p_m d\mu_n - \int f d\mu_n\right|$$

For the first part, $|\int fd\mu - \int p_m d\mu| \leq \|f - p_m\|_\infty \mu(X) \to 0$ as $m \to \infty$. Similarly, $\left|\int p_m d\mu_n - \int f d\mu_n\right| \leq \|f - p_m\|_\infty \mu_n(X) \to 0$ for all $n$.

Next, find a polynomial $q_{m_j}$ with degree at most $j$ such that $\|q_{m_j} - p_m\|_\infty \to 0$ as $j \to \infty$. Then since $\mu_n \to \mu$ in moments, then $\left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| \to 0$ for all $j$. Thus,

$$\begin{aligned}\left|\int p_m d\mu - \int p_m d\mu_n\right| &\leq \left|\int p_n d\mu - \int q_{m_j} d\mu\right| + \left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| + \left|\int q_{m_j} d\mu_n - \int p_m d\mu_n\right|\\ &\leq \|p_m - q_{m_j}\|_\infty (\mu(X) + \mu_n(X)) + \left|\int q_{m_j} d\mu - \int q_{m_j} d\mu_n\right| \to 0.\end{aligned}$$

□

# 16 January 2012

**Problem 1.** *Let $\mathcal{A}$ be the subset of $[0,1]$ consisting of numbers whose decimal expansions contain no sevens. Show that $\mathcal{A}$ is Lebesgue measurable, and find its measure. Why does non-uniqueness of decimal expansions not cause any problems?*

*Proof.* Let $A_i$ be the subset of $[0,1]$ consisting of numbers whose first $i$ digits are not 7. Then $A_{n+1} \subseteq A_n$ and $A = \cap_n A_n$,

$$A_1 = [0, 0.7] \cup [0.8, 1]$$
$$A_2 = [0, 0.07] \cup [0.08, 0.17] \cup \ldots \cup [0.98, 1]$$

So $A_n$ is the union of some Borel intervals in $[0,1]$, so $A_n$ is Lebesgue measurable. Therefore, $A$ is Lebesgue measurable.

Now for $0 \le i \le 9$, let $A_n^i$ be the subset of $A_n$ such that the $(n+1)$th digit is $i$. Then we can write $A_n = \sqcup_{i=0}^9 A_n^i$.

Also, $m(A_n^i) = m(A_n^j)$, so $m(A_n) = 10m(A_n^i)$ and $A_{n+1} = \sqcup_{i \neq 7} A_n^i$ so $m(A_{n+1}) = 9m(A_n^i)$. Therefore, $m(A_{n+1}) = \frac{9}{10} m(A_n)$. Then

$$m(A) = m\left(\cap_{n=1}^\infty A_n\right) = \lim_n m(A_n) = \lim_n \left(\frac{9}{10}\right)^{n-1} m(A_1) = 0$$

The only numbers with non-unique decimal representation are $0.a_1 a_2 \ldots a_n = 0.a_1 a_2 \ldots a_{n-1} 999\ldots$. However $\forall n$ there are only finitely many, so non-unique $= \cup_n \{0.a_1 \ldots a_n\}$ which is countable, hence null, hence Lebesgue. ☐

**Problem 2.** *Let the functions $f_\alpha$ be defined by*

$$f_\alpha(x) = \begin{cases} x^\alpha \cos(1/x) & x > 0 \\ 0 & x = 0 \end{cases}$$

*Find all values of $\alpha \ge 0$ such that*

*(a) $f_\alpha$ is continuous*

*Proof.* When $a > 0$, $x^a \cos(1/x) \le x^a \to 0$ as $x \to 0$ so $f_a$ is continuous. If $a = 0$, we know $\cos(1/x)$ isn't continuous at 0. ☐

*(b) $f_\alpha$ is of bounded variation on $[0,1]$*

*Proof.* First, $0 < a \le 1$, put partitions

$$P_m = \left\{0, \frac{1}{2\pi m}, \frac{1}{\pi(2m-1)}, \ldots, \frac{1}{\pi}, 1\right\}$$

Then

$$f_a(P_m) = \left\{0, \frac{1}{(\pi 2m)^a}, \frac{-1}{(\pi(2n-1))^a}, \ldots, \frac{-1}{\pi^a}, \cos(1)\right\}$$

So

$$T_{f_\alpha}(P_m) = \left|\frac{1}{(\pi(2m))^a} - 0\right| + \left|\frac{-1}{(\pi(2m-1))^a} - \frac{1}{(\pi 2m)^a}\right| + \ldots + \left|\cos(1) - \frac{-1}{\pi^a}\right| \approx \sum_{i=1}^{2m} \frac{c}{(\pi i)^a} \to \infty$$

when $0 < a \leq 1$ as $m \to \infty$.

So when $0 < a \leq 1$, $f_a$ is not of bounded variation when $0 < a \leq 1$. For $a > 1$, let's look at (c). □

*(c) $f_\alpha$ is absolutely continuous on* $[0, 1]$

*Proof.* When $a > 1$, we see $f_a'(0) = 0$ and $f_a'$ is integrable because $f_a'(x) = ax^{a-1}\cos(1/x) + x^{a-2}\sin(1/x)$ so then $f_a(x) = \int_0^x f_a'(t)dt$. Thus, $f$ is absolutely continuous.

So in (b) we have $f_a$ is of bounded variation for $a > 1$. Since $f_a$ isn't bounded variation when $0 < a \leq 1$, so $f_a$ isn't absolutely continuous either when $0 < a \leq 1$. □

**Problem 3.** *Let $\mathcal{F}$ denote the family of functions on* $[0, 1]$ *of the form*

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$$

*where $a_n$ are real and $|a_n| \leq 1/n^3$. State a general theorem and use that theorem to prove taht any sequences in $\mathcal{F}$ has a subsequence that converges uniformly on* $[0, 1]$.

*Proof.* We'll use Arzela-Ascoli.

For all $f \in \mathcal{F}$,

$$|f(x)| = \left|\sum_{n=1}^{\infty} a_n \sin(nx)\right| \leq \sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} n^{-3} < \infty$$

so uniformly bounded. Also, for all $f \in \mathcal{F}$,

$$|f(x)-f(y)| \leq \sum_{n=1}^{\infty} |a_n||\sin(nx)-\sin(ny)| \leq \sum_{n=1}^{\infty} 2n^{-3}\left|\cos\frac{nx+ny}{2}\right|\left|\sin\frac{nx+ny}{2}\right| \leq \sum_{n=1}^{\infty} n^{-2}|x-y| = \frac{\pi^2}{6}|x-y|$$

So $\mathcal{F}$ is equicontinuous.

Then $\overline{\mathcal{F}}$ is compact, hence sequentially compact. So $\mathcal{F}$ has a subsequence that converges in the uniform norm. □

**Problem 4.** *Let $\mathcal{H}$ be a Hilbert space and $W \subset \mathcal{H}$ a subspace. Show that $\mathcal{H} = \overline{W} \oplus W^{\perp}$ where $\overline{W}$ is the closure of $W$.*

*Note: Do not just state this as a consequence of a standard result, prove the result.*

*Proof.* here! □

**Problem 5.** *Suppose $A$ is a bounded linear operator on a Hilbert space $\mathcal{H}$ with the property that*

$$\|p(A)\| \leq C \sup\{|p(z)| \mid z \in \mathbb{C}, |z| = 1\}$$

*for all polynomials $p$ with complex coefficients, and a fixed constant $C$. Show that to each pair $x, y \in \mathcal{H}$ there corresponds a complex Borel measure $\mu$ on the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ such that*

$$\langle A^n x, y\rangle = \int z^n d\mu(z) \qquad n = 0, 1, 2, \dots$$

*Proof.* Consider

$$\begin{aligned} T_{x,y} : P(S^1) &\to \mathbb{C} \\ p &\mapsto \langle P(A)x, y\rangle \end{aligned}$$

Then

$$|\langle P(A)x, y\rangle| \leq \|P(A)\|\|x\|\|y\| \leq C\|P\|_\infty \|x\|\|y\|$$

Thus, $|T_{x,y}(P)| \leq C\|x\|\|y\|\|P\|_\infty = f(P)$ which is obviously a seminorm. By Hahn-Banach, $T_{x,y}$ can be extended to $C(S^1)$.

Then apply Riesz-Representation Theorem, there exists a complex Borel measure $\mu$ on $S^1$ such that

$$T_{x,y}(P) = \langle P(A)x, y\rangle = \int_{S^1} P(z) d\mu(z)$$

Take $P(z) = z^n$ so $\langle A^n x, y\rangle = \int_{S^1} z^n d\mu(z)$. □

**Problem 6.** *Let $\phi$ be the linear functional*

$$\phi(f) = f(0) - \int_{-1}^{1} f(t) dt$$

*(a) Compute the norm of $\phi$ as a functional on the Banach space $C[-1, 1]$ with uniform norm*

*Proof.*
$$|\phi(f)| \leq |f(0)| + \int_{-1}^{1} |f(t)|dt \leq \|f\|_\infty + \|f\|_\infty \int_{-1}^{1} dt = 3\|f\|_\infty$$

So $\|\phi\| \leq 3$. On the other hand, let $f_n$ be piecewise linear functional such that $f_n = -1$ on $[-1, -1/n]$ and $[1/n, 1]$ and $f_n(0) = 1$. Then

$$\int_{-1}^{1} f_n(t)dt = -2(1 - 1/n) + \frac{2}{n} = -2 + \frac{4}{n} \to -2$$

So $\sup |\phi(f_n)| \geq 3$ so $\|\phi\| = 3$. □

(b) *Comptue the norm of $\phi$ as a functional on the normed vector space $LC[-1,1]$ which is $C[-1,1]$ with the $L^1$ norm.*

*Proof.*
$$\|\phi\| = \sup_{f \in LC[-1,1]} \frac{|f(0) - \int_{-1}^{1} f(t)dt|}{\|f\|_1} \geq \lim_n \frac{|1 - 1/n|}{(1/n)} = \infty$$

□

**Problem 7.** *Let $X$ be a normed space and $\mathcal{A} \subset X$ be a subset. Show that $\mathcal{A}$ is bounded (as a set) if and only if it is weakly bounded (that is, $f(A) \subset \mathbb{C}$ is bounded for each $f \in X^*$).*

*Proof.* $\Rightarrow$) for all $x \in A$, for all $f \in X^*$, $|f(x)| \leq \|f\|\|x\| < \infty$ so $A$ is weakly bounded

$\Leftarrow$) on the other hand, consider $A^{**} = \{a^{**} \mid a \in A\}$ by $a^{**}(f) = f(a)$ for all $f \in X^*$. Since $X^*$ is Banach, and we know

$$\sup_{a^{**} \in A^{**}} \|a^{**}(f)\| = \sup_{a \in A} |f(a)| < \infty \qquad \forall f \in X^*$$

By the uniform boundedness principle, $\sup_{a \in A} \|a\| = \sup_{a^{**} \in A^{**}} \|a^{**}\| < \infty$. □

**Problem 8.** *Let $X$ be a topological vector space.*

(a) *Define what this means.*

*Proof.* Let $X$ be a vector space, $\mathcal{T}$ a topology on $X$. Then $(X, \mathcal{T})$ is a topological vector space provided

- $+ : X \times X \to X$ is continuous
- $\cdot : \mathbb{R} \times X \to X$ is continuous

□

(b) *Let $A \subset X$ be compact and $B \subset X$ be closed. Show that $A + B \subset X$ is closed.*

*Proof.* Fix $z \in (A+B)^c$. For $x \in A$, $z - x \in B^c$ so there exists an open neighborhood $V_x \ni 0$ in $X$ such that $(z - x + V_x) \cap B = \emptyset$. Since addition is continuous, there exists $U_{1x}, U_{2x}$ neighborhoods of 0 such that $U_{1x} + U_{2x} \subseteq V_x$.

$U_x = U_{1x} \cap U_{2x} \cap (-U_{1x}) \cap (-U_{2x})$ so $U_x = -U_x$. Then $\{x + U_x\}_{x \in A}$ is an open cover of $A$. Since $A$ is compact, there exists a finite subcover $x_1, \ldots, x_n \in A$ such that

$$A \subseteq \cup_{i=1}^n x_i + U_{x_i}$$

Put $U = \cap_{i=1}^n U_{x_i}$. Then $z + U$ is an open neighborhood of $z$. If there exists $x \in A$ $y \in B$ such that $x + y \in z + U$ then $x \in x_i + U_{x_i}$ for some $i$ and $y \in z - x + U \subseteq z - x_i + U_{x_i} \subseteq z - x_i + V_{x_i}$ but $(z - x_i + V_{x_i}) \cap B = \emptyset$. Contradiction!

So $(z + U) \cap (A + B) = \emptyset$. □

(c) *Give an example indicating that the condition 'A closed' is insufficient for the conclusion.*

*Proof.* $X = \mathbb{R}^2$, $A = \{(x, 0) \mid x \in \mathbb{R}\}$ and $B = \{(x, 1/x) \mid x > 0\}$. Then $A + B = \{(x, y) \mid y > 0\}$. □

**Problem 9.** *Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $f, f_n \in L^3(X, d\mu)$ for $n \in \mathbb{N}$ be functions such that $f_n \to f$ $\mu$-a.e. and $|f_n| \leq M$ for all $n$. Let $g \in L^{3/2}(X, d\mu)$. Show that*

$$\lim_n \int f_n g d\mu = \int f g d\mu.$$

*Proof.* $|f_n g| \leq M|g|$. Since $\mu$ is a finite measure, $M1 \in L^3(\mu)$. By Holder, $M|g| \in L^1(\mu)$. The result follows from Dominated Convergence Theorem. □

**Problem 10.** *Let $X$ be a $\sigma$-finite measure space, and $f_n : X \to \mathbb{R}$ a sequence of measurable functions on it. Suppose $f_n \to 0$ in $L^2$ and $L^4$.*

(a) *Does $f_n \to 0$ in $L^1$?*

*Proof.* NO.

Let $X = \mathbb{R}$, $\mu$=Lebesgue measure. $f_n = n^{-1}\chi_{[0,n]}$ so $\|f_n\|_1 = 1$ does not converge to 0, but $\|f_n\|_2 = n^{-1/2} \to 0$ and $\|f_n\|_4 = n^{-3/4} \to 0$. □

(b) *Does $f_n \to 0$ in $L^5$?*

*Proof.* YES.

Since $0 < 2 < 3 < 4 < \infty$, $L_2 \cap L_4 \subseteq L_3$ and $\|f\|_3 \leq \|f\|_2^\lambda \|f\|_4^{1-\lambda}$ where $\frac{1}{3} = \frac{\lambda}{2} + \frac{1-\lambda}{4}$ implies $\lambda = \frac{1}{3}$. So

$$\|f_n\|_3 \leq \|f_n\|_2^{1/3} \|f_n\|_4^{2/3} \to 0$$

□

(c) *Does $f_n \to 0$ in $L^5$?*

*Proof.* NO.

$X = [0,1]$, $\mu$: Lebesgue measure. Let $f_n = n\chi_{[0,n^{-5}]}$. Then $\|f_n\|_5 = 1$ but $\|f_n\|_2 = n^{-3/2} \to 0$ and $\|f_n\|_4 = n^{-1/4} \to 0$. □

# 17 August 2011

**Problem 1.** *Let $(X, \mathcal{M}, \mu)$ be a measure space.*

(a) *Give the definitions of convergence a.e. and convergence in measure for a sequence of measurable functions on $X$.*

*Proof.* We say a sequence of measurable functions $f_n$ converge to $f$ almost everywhere if $\mu(\{x \mid \lim_n f_n(x) \neq f(x)\}) = 0$.

We say that $f_n$ converges to $f$ in measure if $\forall \epsilon > 0$, $\lim_n \mu(\{x \mid |f(x) - f_n(x)| > \epsilon\}) = 0$. □

(b) *Show that every sequence of measurable functions on $X$ which converges in measure to 0 has a subsequence which converges a.e. to 0.*

*Proof.* Suppose for every $\epsilon > 0$, $\mu(\{x \mid |f_n(x)| \geq \epsilon\}) \to 0$. Choose a subsequence $\{f_{n_k}\}$ such that if

$$E_j = \{x \mid |f_{n_j}(x) - f_{n_{j+1}}(x)| > 2^{-j}\}$$

satisfies $\mu(E_j) < 2^{-j}$. Let $F_k = \cup_{j=k}^{\infty} E_j$ so $\mu(F_k) \leq \sum_{j=k}^{\infty} 2^{-j} \leq 2^{1-k}$. Let $F = \cap_k F_k$ so $\mu(F) = 0$.

For $x \notin F_k$ and for $i \geq j \geq k$ then

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{\ell=j}^{i-1} |f_{n_\ell}(x) - f_{n_{\ell+1}}(x)| \leq \sum_{\ell=j}^{i-1} 2^{\ell} \leq 2^{-j} \to 0 \quad \text{as } k \to \infty.$$

So $f_{n_k}$ is pointwise Cauchy on $x \notin F$, so let

$$f(x) = \begin{cases} \lim f_{n_k}(x) & x \notin F \\ 0 & \text{otherwise} \end{cases}$$

So $f_{n_k} \to 0$ almost everywhere and $f_n \to f$ in measure since

$$\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f_n(x) - f_{n_\ell}(x)| \geq \epsilon/2\})}_{\to 0} + \underbrace{\mu(\{x \mid |f_{n_\ell}(x) - f(x)| \geq \epsilon\})}_{\to 0}$$

and

$$\mu(\{x \mid |f(x)| \geq \epsilon\}) \leq \underbrace{\mu(\{x \mid |f(x) - f_n(x)| \geq \epsilon/2\}}_{\to 0} + \underbrace{\mu(\{x \mid |f_n(x)| \geq \epsilon/2\})}_{\to 0}$$

so $f = 0$ almost everywhere. Thus, $\{f_{n_k}\}$ converges to 0 almost everywhere. □

**Problem 2.** *Let $X$ be a separable Banach space. Show that there exists an isometric linear map from $X$ into $\ell^\infty$. Also, show that this is false in general if $\ell^\infty$ is replaced by $\ell^2$.*

*Proof.* Let $(x_n)$ be a dense sequence in $B_X$. For each $n$, use Hahn-Banach Theorem to find a norm-one functional $f_n \in X^*$ with $f_n(x_n) = 1$.

Define $\phi : X \to \ell^\infty$ via $\phi(x) = \big(f_n(x)\big)$. Suppose $x \in X$ has norm one and let $1 > \epsilon > 0$. Choose $n_\epsilon$ so that $\|x_{n_\epsilon} - x\| < \epsilon$. Then

$$\epsilon > |f_{n_\epsilon}(x_{n_\epsilon} - x)| = |f_{n_\epsilon}(x)|$$

So $\|\phi(x)\| = \sup_n |f_n(x)| \geq 1$. For every $n$, $|f_n(x)| \leq \|f_n\|\|x\| = 1$ so $\|\phi(x)\| \leq 1$. So $\|\phi(x)\| = 1$ whenever $\|x\| = 1$. Then for all non-zero $x$, $\|\phi(x)\| = \|x\| \sup_n |f_n(x/\|x\|)| = \|x\|$. So $\phi$ is an isometry.

Why FALSE for $\ell^2$? □

**Problem 3.** *Let $X$ be a locally compact metric space and let $\{x_k\}$ be a sequence in $X$ which has no convergent subsequence. Show that $\{n^{-1}\sum_{k=1}^n \delta_{x_k}\}$ converges to 0 in the weak* topology on $C_0(X)^*$, where $\delta_{x_k}$ denotes the point mass at $x_k$.*

*Proof.* here □

**Problem 4.** *Let $\mathcal{P}$ be the set of all polynomials $f$ on $[0,1]$ such that $f(0) = f'(0) = 0$. Determine, with proof, the values of $p$ with $1 \leq p \leq \infty$ such that $\mathcal{P}$ is dense in $L^p[0,1]$.*

*Proof.* All $1 \leq p < \infty$. Clearly, $\mathcal{P}$ is an algebra which separates points (ex. $x^2$). Stone-Weierstrass implies $\overline{\mathcal{P}} = \{f \in C[0,1] \mid f(0) = 0\}$. Now for any $f \in L^p$, for all $\epsilon > 0$, there exists some $N$ such that

$$\left\|f - f\chi_{[-N \leq f \leq N]}\right\|_p \leq \frac{\epsilon}{2}$$

Define $f_N = f\chi_{[-N \leq f \leq N]}$. By Lusin's theorem, there exists a closed set $F$ such that $m([0,1]\backslash F) \leq \frac{1}{2^p}\frac{\epsilon^p}{(2N)^p} = \frac{1}{2^p}\left(\frac{\epsilon}{2N}\right)^p$. and $f_N|_F$ continuous.

Tietze extension theorem applied to $f_N$ and $F$ implies the extension $g$ is still bounded by $N$. Then

$$\|f_N - g\|_p^p = \int_{[0,1]\backslash F} |f_N - g|^p \leq (2N)^p m([0,1]\backslash F) \leq \frac{\epsilon^p}{2^p}$$

So then

$$\|f-g\|_p \le \|f-f_N\|_p + \|f_N - g\|_p \le \frac{\epsilon^p}{2^p} + \frac{\epsilon}{2} \le \epsilon$$

NOT $L^\infty[0,1]$ since $\overline{\mathcal{P}} = \{f \in C[0,1] \mid f(0)=0\}$. If $f \in L^\infty[0,1]$ with $f(0) = a \neq 0$, then $\forall g \in \overline{\mathcal{P}}$, $\|f-g\|_\infty = a$. □

**Problem 5.** *Let $1 < p < \infty$ and let $\{x_k\}_{k=1}^\infty$ be a sequence in $\ell^p(\mathbb{N})$ such that $\lim_k x_k(n) = 0$ for all $n \in \mathbb{N}$. Show that if there is an $M > 0$ such that $\|x_k\| \le M$ for all $k \in \mathbb{N}$ then $x_k \to 0$ weakly.*

*Also, show that if no such $M$ exists, then $\{x_k\}$ can fail to converge weakly.*

*Proof.* Note: Similar to August 2015, #3, just in a different space now.

Fix some $y \in \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. We want to show that $\sum_n x_k(n)y(n) \to 0$ as $k \to \infty$. Fix $\epsilon > 0$.

Then we may choose a finite $A \subseteq \mathbb{N}$ such that $\sum_{A^c} |y(n)|^q < \epsilon^q$. Since $A$ is finite, choose some $K$ such that for all $k \ge K$ we have $|x_k(n)|^p < \frac{\epsilon^p}{|A|}$. Then for all $k \ge K$, by using Holder, we have

$$\begin{aligned}
|y(x_k)| &\le \sum_{n\in\mathbb{N}} |x_k(n)||y(n)| \\
&= \sum_{n\in A} |x_k(n)||y(n)| + \sum_{n \in A^c} |x_k(n)||y(n)| \\
&\le \left(\sum_{n\in A} |x_k(n)|^p\right)^{1/p} \left(\sum_{n\in A} |y(n)|^q\right)^{1/q} + \left(\sum_{n\in A^c} |x_k(n)|^p\right)^{1/p} \left(\sum_{n\in A^c} |y(n)|^q\right)^{1/q} \\
&\le \left(|A| \frac{\epsilon^p}{|A|}\right)^{1/p} \|y\|_q + M\epsilon \\
&= \epsilon\left(\|y\|_q + M\right)
\end{aligned}$$

By making $\epsilon$ small enough, we see that $|y(x_k)| \to 0$ as $k \to \infty$.

To see why we require $(x_k)$ to be bounded, consider $p = q = 2$. Take

$$x_k = (0, 0, \ldots, 0, 2^k, 0, \ldots) = 2^k e_k \qquad y = \left(\frac{1}{2}, \frac{1}{2^2}, \ldots, \frac{1}{2^n}, \ldots\right)$$

where $x_k$ is all zeros except in the kth spot. Then we can see that $\lim_k x_k(n) = 0$ for all $n$, but that for all $k$,

$$y(x_k) = \sum_n x_k(n)y(n) = 2^k \frac{1}{2^k} = 1$$

□

**Problem 6.** *Let $f \in C_0(\mathbb{R})$ and for every $t \in \mathbb{R}$ define $f_t \in C_0(\mathbb{R})$ by $f_t(x) = f(x+t)$ for all $x \in \mathbb{R}$.*

*(a) Prove that $\{f_t \mid t \in [0,1]\}$ is compact in the norm topology.*

*Proof.* Similar to August 2013 #1

Since $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, we can choose $g \in C_c^\infty(\mathbb{R})$ such that $\|g - f\|_\infty < \epsilon$. Then

$$\|f_{t_n} - f_t\|_\infty \leq \|f_{t_n} - g_{t_n}\|_\infty + \|g_{t_n} - g_t\|_\infty + \|g_t - f_t\|_\infty$$

It's easy to see $\|f_{t_n} - g_{t_n}\|_\infty$ and $\|g_t - f_t\|_\infty$ are small since $\|g - f\|_\infty < \epsilon$. For $\|g_{t_n} - g_t\|_\infty$, then

$$\|g_{t_n} - g_t\|_\infty = \sup_{x\in\mathbb{R}} |g_{t_n}(x) - g_t(x)| = \sup_{x\in\mathbb{R}} |g(x + t_n) - g(x + t)|$$

where for each fixed $t_n \to t$, since $g$ is compactly supported and continuous then can be sufficiently small for large enough $n$.

Therefore, the map $G : \mathbb{R} \to C_0(\mathbb{R})$ given by $G(t) = f_t$ is continuous. Since $\{f_t \mid t \in [0, 1]\} = G([0, 1])$ and continuous maps preserve compactness, then the set is compact in the norm topology. □

*(b) Prove that $\{f_t \mid t \in \mathbb{R}\}$ is relatively compact in the weak topology.*

*Proof.* here □

**Problem 7.** *Let $f$ be an arbitrary real valued function on $[0, 1]$. Show that the set of points at which $f$ is continuous is a Lebesgue measurable set.*

*Proof.* Similar to August 2016, #3.

In fact, we will prove that the set of points at which $f$ is discontinuous is a countable union of closed subsets.

$f$ is continuous at $p$ if for all $n$, there exists an open $U$ containing $p$ such that $|f(x) - f(y)| < 1/n$ for all $x, y \in U$. Fix $n$ and let

$$V_n = \bigcup_p \{p \text{ s.t. there exists an appropriate } U\} = \bigcup \{\text{appropriate } U\}$$

Hence, $V_n$ is open. Then

$$\{\text{points where } f \text{ is continuous}\} = \bigcap_n V_n$$

So $\{\text{points where } f \text{ is discontinuous}\} = \bigcup_n V_n^c$ where $V_n^c$ is closed. □

**Problem 8.** *Show that not every nonempty bounded closed subset of $\ell^2$ has a point of minimal norm, but that every nonempty bounded closed convex subset of $\ell^2$ has a point of minimal norm.*

*Proof.* Let $C$ be the bounded, closed, convex subset of $\ell^2$. Consider the set $\{y \in \mathbb{R} \mid y = \|x\|, x \in C\}$ and since this set is bounded below, there exists an infimum of the set, say $s$. Then we can find a sequence $x_n \in C$ such that $s \leq \|x_n\| \leq s + \frac{1}{n}$.

I claim that $(x_n)$ is a Cauchy sequence. Indeed, for any $\epsilon > 0$, choose $r$ to be the positive root of the equation $r^2 + 2rs - \frac{\epsilon^2}{4} = 0$.

Since $\|x_n\| \to s$ then there is an $N$ such that $s \leq \|x_n\| < s + r$ for all $n \geq N$. If $n, m \geq N$, then

$$\left\|\frac{x_m - x_n}{2}\right\|^2 = 2\left\|\frac{x_m}{2}\right\|^2 + 2\left\|\frac{x_n}{2}\right\|^2 - \left\|\frac{x_m + x_n}{2}\right\|^2 < \frac{(s+r)^2}{2} + \frac{(s+r)^2}{2} - s^2 = 2sr + r^2 = \frac{\epsilon^2}{4}.$$

So $(x_n)$ is a Cauchy sequence, which means it converges to some $x$. Since $C$ is closed, $x \in C$ and obtains minimal norm.

*Note:* This choice of $x$ is unique! If there were two points of minimal norm, say $x_1$ and $x_2$ then $\frac{1}{2}(x_1 + x_2) \in C$ by the convexity of $C$. So $s \leq \left\|\frac{1}{2}(x_1 + x_2)\right\| \leq \frac{1}{2}\|x_1\| + \frac{1}{2}\|x_2\| = s$. Hence, $\|x_1 + x_2\| = 2s$. By the parallelogram law,

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 = 2\|x_1\|^2 + 2\|x_2\|^2$$

And so $\|x_1 - x_2\|^2 = 4s^2 - 4s^2 = 0$ so $x_1 = x_2$, proving uniqueness.

Counterexample: Consider $M = \left\{\frac{n+1}{n} e_n \mid n \in \mathbb{N}\right\}$. $M$ is closed since the distance between any two of its elements is greater than $\sqrt{2}$ (and thus the only convergent sequences from $M$ are those that are eventually constant). $M$ is clearly non-empty and has no element of minimal norm. □

**Problem 9.** *Show that there is a sequence $\{f_n\}$ of continuous functions on $[0,1]$ such that*

*(a) $|f_n(t)| = 1$ for all $n$ and all $t \in [0,1]$ and*

*(b) for all $g \in L^1[0,1]$ one has $\int_0^1 f_n(t)g(t)dt \to 0$ as $n \to \infty$*

*Proof.* NOT POSSIBLE??

If $f_n$ is continuous on $[0,1]$ and $|f_n(t)| = 1$ then $f_n(t) = \pm 1$. Since $f_n$ is continous, each $f_n$ is the constant function at either 1 or $-1$. Write it as $f_n(x) = (-1)^{k_n}$ where $k_n$ is even or odd depending on $n$.

Then if we take $g$ to be the constant function 1, we get

$$\int_0^1 f_n(t)g(t)dt = \int_0^1 (-1)^{k_n} dt = (-1)^{k_n}$$

which does not have to converge to 0 as $n \to \infty$.

right? obvious? I don't get it... □

**Problem 10.** *(a) Define what it means for a real valued function on $[0,1]$ to be absolutely continuous.*

*Proof.* The function $f : [0,1] \to \mathbb{R}$ is absolutely continuous if for every $\epsilon > 0$ there exists $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $[0,1]$ with $x_k, y_k \in [0,1]$ satisfy $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \epsilon$.

Equivalently, $f$ has a derivative $f'$ almost everywhere and the derviative is Lebesgue integrable and for all $x \in [0,1]$,

$$f(x) = f(0) + \int_0^x f'(t)dt.$$

□

(b) *Prove that if* $f$ *and* $g$ *are absolutely continuous strictly positive functions on* $[0,1]$ *then* $f/g$ *is absolutely continuous on* $[0,1]$.

*Proof.* Step 1: If $f$ is absolutely continuous, then so is $1/f$.

Since $f > 0$ is continuous on a compact space, there exists some $M \in \mathbb{N}$ such that $\frac{1}{M} \leq |f(x)| \leq M$ for all $x \in [0,1]$.

Indeed, since $g$ is absolutely continuous then for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever a finite sequence of pairwise disjoint sub-intervals $(x_k, y_k)$ of $[0,1]$ with $x_k, y_k \in [0,1]$ satisfy $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(y_k) - f(x_k)| < \frac{\epsilon}{M^2}$. Then for such intervals, we have

$$\begin{aligned}
\sum \left| \frac{1}{f(y_k)} - \frac{1}{f(x_k} \right| &= \sum \left| \frac{f(x_k) - f(y_k)}{f(y_k)f(x_k)} \right| \\
&\leq \sum \left| \frac{1}{f(y_k)} \right| \left| \frac{1}{f(x_k)} \right| |f(y_k) - f(x_k)| \\
&\leq M^2 \sum |f(y_k) - f(x_k)| \\
&= M^2 \frac{\epsilon}{M^2} = \epsilon.
\end{aligned}$$

Step 2: If $f$ and $g$ are both absolutely continuous, then so is $fg$.

Find $M \in \mathbb{N}$ such that $|f(x)|, |g(x)| \leq M$ for all $x \in [0,1]$.

Take $\delta_1$ such that if $\sum y_k - x_k < \delta_1$ then $\sum |f(y_k) - f(x_k)| < \epsilon/2M$. Similarly, take $\delta_2$ such that if $\sum y_k - x_k < \delta_2$ then $\sum |g(y_k) - g(x_k)| < \epsilon/2M$. Let $\delta = \min(\delta_1, \delta_2)$. Now

$$\begin{aligned}
\sum |(fg)(y_k) - (fg)(x_k)| &= \sum |f(y_k)g(y_k) - f(x_n)g(x_n)| \\
&\leq \sum |f(y_k)g(y_k) - f(y_k)g(x_k)| + |f(y_k)g(x_k) - f(x_k)g(x_k)| \\
&\leq \sum |f(y_k)||g(y_k) - g(x_k)| + \sum |g(x_n)||f(y_k) - f(x_k)| \\
&\leq M \sum |g(y_k) - g(x_k)| + M \sum |f(y_k) - g(x_k)| \\
&\leq M \frac{\epsilon}{2M} + M \frac{\epsilon}{2M} \\
&= \epsilon
\end{aligned}$$

Combining the two steps, we see immediately that $f/g$ is absolutely continuous. □