Abstract Algebra Test Review 1

Niraj, Arrita, and Kellan

1. External Direct Products
   - Definitions and properties (order of an element, know when is $G \oplus H$ cyclic)
   - Be able to view $U(n)$ as an external direct product
2. Normal Subgroups and Factor Groups
   - Normal subgroups (definition, normal subgroup test)
   - Factor groups (definition, G/Z theorem, $G/Z(G) \approx Inn(G)$ theorem, Cauchy's theorem for Abelian groups)
   - Internal direct products (definitions, IDP $\approx$ EDP, groups of order p^2)
3. Group Homomorphisms
   - Definition and properties (properties of elements/subgroups under homomorphism, $\ker\phi \lhd G$, 1st isomorphism theorem and corollary, normal subgroups are kernels)
4. Fundamental Theorem of Finite Abelian Groups
   - Every finite Abelian group is a direct prod of cyclic groups of prime-power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group (page 226).
   - Be able to find all Abelian groups up to isomorphism
5. Introduction to Rings
   - Definition and properties of a ring, subring test
   - Rules of multiplication
6. Integral Domains
   - Definitions: zero-divisor, ID, fields
   - Relationship between IDs and fields
   - Characteristic of a ring (rings with unity and ID)

Open-ended questions:

1. Prove/disprove $\mathbb{Q}^*$ under multiplication isomorphic to $\mathbb{R}^*$ under multiplication.
2. Prove/disprove $\text{Inn}(G) \triangleleft \text{Aut}(G)$.
3. Prove/disprove that normal groups are closed under intersection.
4. Suppose that $\phi : \mathbb{Z}_8 \to \mathbb{Z}_{20}$ is a group homomorphism with $\phi(3) = 15$.
   - Determine $\phi(x)$.
   - Determine the image and kernel of ϕ
   - Determine $\phi^{-1}(5)$.
5. Prove that the center of a ring is a subring.
6. Let R be a noncommutative ring and let $Z(R)$ be the center of R. Prove that the additive group of $R/Z(R)$ is not cyclic.
7. Find all the Abelian groups of order 200, up to isomorphism.

True, sometimes true, false questions:

1. In a factor group G/H, if $aH = bH$, then $|a| = |b|$.
2. If $H \approx K$, then $G/H \approx G/K$.
3. Let $R_1, R_2, ..., R_n$ be commutative rings with unity. Then $U(R_1 \oplus R_2 \oplus \cdots \oplus R_n) = U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)$.
4. In an integral domain, for distinct positive integers m and n, if $a^m = b^m$ and $a^n = b^n$, then $a = b$.