Solutions to Assignment-2

Only submit the questions in red.

1. (a) For any two sequences $\{a_n\}$ and $\{b_n\}$ show that

$$\limsup_{n\to\infty}(a_n+b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

unless the right hand side is of the form $\infty - \infty$.

Solution: Assume both the limsups are finite (the other cases are also similar). Let $A = \limsup_{n\to\infty} a_n$, $B = \limsup_{n\to\infty} b_n$ and $L = \limsup_{n\to\infty}(a_n + b_n)$. Suppose $L > A + B$. Choose an $\varepsilon > 0$ such that $L - \varepsilon > A + B + \varepsilon$. For any $N > 0$ **there exists** an $n > N$ such that

$$a_n + b_n > L - \varepsilon. \tag{0.1}$$

On the other hand, there exists N_1 such that for all $n > N_1$,

$$a_n < A + \frac{\varepsilon}{2},$$

and there exists N_2 such that for all $n > N_2$,

$$b_n < B + \frac{\varepsilon}{2}.$$

But then if $N = \max(N_1, N_2)$, then **for any** $n > N$ we have

$$a_n + b_n < A + B + \varepsilon < L - \varepsilon,$$

contradicting (0.1).

(b) Find sequences $\{a_n\}$ and $\{b_n\}$ with strict inequality above.

Solution: Let $a_n = (-1)^n$ and $b_n = (-1)^{n-1}$. Then $a_n+b_n = 0$ for all n, and so $\limsup_{n\to\infty}(a_n+b_n) = 0$ while, $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n = 1 + 1 = 2$.

2. Let $\{a_n\}$ be a sequence of real numbers, and let

$$S = \{x \in \mathbb{R} \mid \exists \text{ a sub-sequence } a_{n_k} \text{ such that } a_{n_k} \xrightarrow{k\to\infty} x\}.$$

(a) Show that $L = \limsup a_n$ if and only if $L = \sup S$.

Solution: Suppose $L = \limsup a_n$. First, we claim that $L \in S$. To see this, note that by the equivalent characterization of limsup, there exists n_1 such that

$$a_{n_1} > L - 1.$$

Given n_1, there exists $n_2 > n_1$ such that

$$a_{n_2} > L - \frac{1}{2}.$$

Having chosen $n_1 < n_2 < \cdots < n_{k-1}$, let $n_K > n_{k-1}$ such that

$$a_{n_k} > L - \frac{1}{k}.$$

Claim. $a_{n_k} \xrightarrow{k\to\infty} L$.
Proof. Let $\varepsilon > 0$. Then there exists N such that for all $n > N$,

$$a_n < L + \varepsilon.$$

Since $n_k \xrightarrow{k\to\infty}$, there exists a K_1 such that for all $k > K_1$, $n_k > N$. In particular, for all $k > K_1$,

$$a_{n_k} < L + \varepsilon.$$

Let K_2 such that $1/K_2 < \varepsilon$. Then by our choice of the subsequence a_{n_k}, for all $k > K_2$,

$$a_{n_k} > L - \frac{1}{k} > L - \frac{1}{K_2} > L - \varepsilon.$$

In particular, if $K = \max(K_1, K_2)$, and $k > K$ then

$$|a_{n_k} - L| < \varepsilon,$$

and hence $a_{n_k} \xrightarrow{k\to\infty} L$. $\square$

This shows that $L \in S$. In particular, $L \leq \sup S$. Suppose $L < \sup S$. Let $\varepsilon > 0$ such that $L + \varepsilon < \sup S$. There exists an N such that for all $n > N$,

$$a_n < L + \varepsilon,$$

and so for any $x \in S$, $x < L + \varepsilon$. Taking sup,

$$\sup S \leq L + \varepsilon,$$

a contradiction. Hence $L = \sup S$.

(b) Formulate and prove the analogous statement for $\liminf$.

Solution: The corresponding statement would be

$$\liminf_{n\to\infty} a_n = \inf S.$$

One can argue as above, or alternately, use the standard trick that if $b_n = -a_n$, then

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} b_n.$$

Note. From now on, you can use the conclusions of this exercise as a theorem. So now, you have a definition of $\limsup$ and two other equivalent characterizations.

3. Find the lim sup and lim inf of the sequence $\{a_n\}$ defined recursively by

$$a_1 = 0,\ a_{2m} = \frac{a_{2m-1}}{2},\ a_{2m+1} = \frac{1}{2} + a_{2m}.$$

Justify your answers with complete proofs.

Solution: To get some intuition, we compute the first few terms of the sequence,

$$a_2 = 0,\ a_3 = \frac{1}{2},\ a_4 = \frac{1}{4},\ a_5 = \frac{3}{4},\ a_6 = \frac{3}{8},\ a_7 = \frac{7}{8},\ a_8 = \frac{7}{16},\ a_9 = \frac{15}{16}.$$

Seeing a pattern, we make the claim -

Claim. $\liminf_{n\to\infty} a_n = \frac{1}{2}$ and $\limsup_{n\to\infty} a_n = 1$.

Proof. The easiest proof is to simply find a formula for the n^{th} term. We claim that

$$a_n = \begin{cases} \frac{2^m-1}{2^m}, & n = 2m+1 \\ \frac{2^{m-1}-1}{2^m}, & n = 2m. \end{cases}$$

We prove this by induction. The base cases $n = 1$ are seen to be true. Suppose the formula is correct for some $n = 2m-1 = 2(m-1)+1$. We then prove the formula for $2m$ and $2m+1$.

$$a_{2m} = \frac{a_{2m-1}}{2} = \frac{2^{m-1}-1}{2^m}.$$

But then again by the recursion formula,

$$a_{2m+1} = \frac{1}{2} + a_{2m} = \frac{1}{2} + \frac{2^{m-1}-1}{2^m} = \frac{2^m-1}{2^m}.$$

Once we have the formula, note that$\{a_{2m+1}\}$ is a increasing to 1 and $\{a_{2m}\}$ is a sequence increasing to 1/2. Then clearly, $u_N = \sup\{a_k \mid k > N\} = 1$, and $l_N = \inf\{a_k \mid k > N\} > \frac{2^{N-1}-1}{2^N}$. Letting $N \to \infty$, we complete the proof of the claim.

4. (a) Let $\{a_n\}$ be a bounded sequence with the property that every convergent subsequence converges to the same limit a. Show that the entire sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$.

Solution: If not, then there exists an $\varepsilon > 0$ and a subsequence $b_k = a_{n_k}$ such that

$$|b_k - a| > \varepsilon$$

for all k. By Bozlano-Weierstrass, since $\{b_k\}$ is bounded, there exists a further sub-sequence b_{k_j} which converges. But $b_{k_j} = a_{n_{k_j}}$ is also a sub-sequence of a_n and since it converges, by the hypothesis, it must converge to a. But by our choice of $\{b_k\}$, $|b_{k_j} - a| > \varepsilon$ for all j, a contradiction.

(b) Now assume that $\{a_n\}$ is a sequence with the property that every subsequence has a further subsequence that converges to the same limit a. Show that the entire sequence $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = a$.

Solution: If not, then there is an $\varepsilon > 0$ and a sub-sequence $b_k = a_{n_k}$ such that $|b_k - a| > \varepsilon$. By hypothesis, b_k has a subsequence, say $\{b_{k_j}\}$, that converges to a. But then

$$\lim_{j\to\infty} |b_{k_j} - a| = 0,$$

which contradicts the fact that $|b_{k_j} - a| > \varepsilon$.

5. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers satisfying

$$|a_{n+1} - a_n| \le \frac{1}{2}|a_n - a_{n-1}|.$$

Show that the sequence converges. **Hint.** Show that the sequence is Cauchy.

Solution: Inductively, we see that for any natural number k,

$$|a_{k+1} - a_k| \le \frac{1}{2^k}|a_1 - a_0|.$$

Now if $m > n$ then by triangle inequality

$$\begin{aligned} |a_m - a_n| &= |a_m - a_{m-1} + a_{m-1} - a_{m-2} + a_{m-2} - \cdots - a_n| \\ &\le |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\ &\le |a_1 - a_0| \sum_{k=n}^{m-1} \frac{1}{2^k} \\ &\le \frac{|a_1 - a_0|}{2^n} \sum_{k=0}^{\infty} \frac{1}{2^k} \\ &\le 2^{-n}|a_1 - a_0|. \end{aligned}$$

Given $\varepsilon > 0$, let N such that $2^{-N}|a_1 - a_0| < \varepsilon$. Then for any $m > n > N$, $|a_m - a_n| < \varepsilon$, and the sequence is Cauchy.

6. Let $S = \{n_1, n_2, \cdots\}$ denote the collection of those positive integers that do not have the digit 0 in their decimal representation. (For example $7 \in S$ but $101 \notin S$). Show that $\sum_{k=1}^{\infty} 1/n_k$ converges. **Note.** This should be a surprising result in that leaving out only a few (but of course still infinite) terms out of the harmonic series, we end up with a series that suddenly converges.

Solution: Consider the one-digit numbers in S, namely $\{1, 2, \cdots, 9\}$. Since each is bigger than one, the sum of reciprocals is

$$1 + \frac{1}{2} \cdots + \frac{1}{9} < 9.$$

Next, consider the two-digit numbers in S. There are 81 of them, and each is bigger than 10, and so the sum of reciprocals satisfies the estimate,

$$\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{19} + \frac{1}{21} + \cdots + \frac{1}{98} + \frac{1}{99} < \frac{81}{10}.$$

In general, consider the subset S_k of numbers in S with k digits, that is numbers between 10^k and 10^{k+1}. The number of such numbers is 9^k. That is because there are k digits, and each digit has 9 options. Moreover, all these numbers are bigger than 10^k, and so

$$\sum_{n \in S_k} \frac{1}{n} < \frac{9^{k+1}}{10^k}.$$

Summing over the reciprocals of numbers with at-most m-digits,

$$\sum_{k=1}^{m}\sum_{n\in S_k}\frac{1}{n} < 9\sum_{k=1}^{m}\frac{9^k}{10^k} < \frac{9}{10}\sum_{k=0}^{\infty}\frac{9^k}{10^k} < \frac{81}{10}\frac{1}{1-\frac{9}{10}} = 81.$$

In particular the partial sums of $\sum_{k=1}^{\infty} 1/n_k$ are bounded by 81 and since the terms in the series are positive by the monotone convergence theorem, the series converges.

7. The *Fibonacci numbers* $\{f_n\}$ are defined by

$$f_0 = f_1 = 1, \text{ and } f_{n+1} = f_n + f_{n-1} \text{ for } n = 1, 2, \cdots.$$

For $n = 1, 2, \cdots$, we also define $r_n = f_{n+1}/f_n$.

(a) Find a formula for r_{n+1} in terms of r_n. Dividing the above recurrence by f_n, we obtain

$$r_n = 1 + \frac{1}{r_{n-1}},$$

or

$$r_{n+1} = 1 + \frac{1}{r_n}.$$

(b) Show that $f_n \geq n$ for all $n \geq 2$.

Solution: Easy proof by induction.

(c) Show that $f_{n+1}f_{n-1} - f_n^2 = (-1)^{n+1}$.

Solution: We proceed by induction. For $n = 1$, $f_{n+1}f_{n-1} - f_n^2 = f_2 f_0 - f_1^2 = 2 \cdot 1 - 1^1 = 1 = (-1)^{1+1}$, and so the identity is verified. Suppose the identity is verified for $n - 1$, that is we have $f_n f_{n-2} - f_{n-1}^2 = (-1)^n$. Then

$$\begin{aligned} f_{n+1}f_{n-1} - f_n^2 &= (f_n + f_{n-1})f_{n-1} - f_n^2 \\ &= f_{n-1}^2 + f_n(f_{n-1} - f_n) \\ &= f_{n-1}^2 - f_n f_{n-2} = -(-1)^n = (-1)^{n+1}. \end{aligned}$$

(d) Hence show that if $n \geq 2$, then

$$|r_{n+1} - r_n| \leq \frac{1}{(n-1)^2}.$$

Solution: Note that

$$|r_n - r_{n-1}| = \left|\frac{f_{n+1}}{f_n} - \frac{f_n}{f_{n-1}}\right| = \frac{1}{f_n f_{n-1}}$$

by the identity. By part (b), $f_n > n$ for all $n \geq 2$ and so the proof is completed by the elementary observation that $(n-1)^2 < n(n-1)$.

(e) Hence show that the sequence of ratios $\{r_n\}$ converge, and compute it's limit. **Note.** This limit is the so-called *golden ratio*.

Solution: For any $n < m$, by the triangle inequality and part(d),

$$|r_m - r_n| \leq \frac{1}{n^2} + \frac{1}{(n+1)^2} + \cdots \frac{1}{(m-1)^2}.$$

Since the right hand side is the tail end of a converging series, by the Cauchy criteria, for any $\varepsilon > 0$, there exists an N such that for any $n, m > N$, the right hand side can be made smaller than ε. This shows that $\{r_n\}$ is Cauchy, and hence converges. To find the actual limit, first note that if $L = \lim_{n\to\infty} r_n$, then $L \neq 0$. letting $n \to \infty$ on both sides of the recurrence obtained in part(a) we obtain

$$L = 1 + \frac{1}{L}.$$

Solving the quadratic $L^2 - L - 1 = 0$, we see that the roots are $(1 \pm \sqrt{5})/2$, of which the only positive root has to be L.

8. Investigate the behavior of each series (convergence, divergence, conditional convergence, absolute convergence). In cases that there is a parameter (p, q or r) find the range of values where the series exhibits the above behavior.

   1. $\sum_{n=1}^{\infty} p^n n^p$ $(p > 0)$
   2. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$
   3. $\sum_{n=1}^{\infty} \frac{1}{p^n - q^n}$, $(0 < q < p)$
   4. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$
   5. $\sum_{n=1}^{\infty} (\sqrt[n]{n} - 1)^n$
   6. $\sum_{n=1}^{\infty} \frac{1}{1+r^n}$.

Solution:

1. Let $a_n = p^n n^p$. Then $\sqrt[n]{a_n} = p n^{p/n} \xrightarrow{n\to\infty} p$. So by the root test, the series converges if $p < 1$ and diverges if $p > 1$. If $p = 1$, the series $\sum n$ clearly diverges.

2. Let $a_n = (-1)^n \frac{\sqrt{n+1}-\sqrt{n}}{n^p}$. Then

$$|a_n| = \frac{1}{n^p(\sqrt{n+1} + \sqrt{n})}.$$

Comparing this with $n^{-(p+\frac{1}{2})}$, we see that $\sum a_n$ converges **absolutely** for $p > 1/2$. On the other hand when $p < 0$, clearly the series diverges. At $p = 0$, the partial sums of the series are $s_n = \sum_{k=1}^{n} \sqrt{k+1} - \sqrt{k} = \sqrt{n+1} - 1$ which clearly diverge. For $p \in (0, 1/2]$ the series does not converge absolutely. To check for conditional convergence we apply alternating series test. Let $b_n = \frac{\sqrt{n+1}-\sqrt{n}}{n^p} = \frac{1}{n^p(\sqrt{n+1}+\sqrt{n})}$, and hence decreases to 0 if $p > 0$. So by the alternating series test the series converges conditionally in the range $p \in (0, 1/2]$.

3. We can write

$$a_n = \frac{1}{p^n - q^n} = \frac{1}{p^n(1 - (q/p)^n)}.$$

Since $q < p$, $\lim_{n\to\infty}(q/p)^n = 0$, and so there exists an N such that for all $n > N$, $(q/p)^n < 1/2$ or $(1 - (q/p)^n)^{-1} < 2$. On the other hand, for any n, $(1 - (q/p)^n)^{-1} > 1$, and so for $n > N$,

$$\frac{1}{p^n} < a_n < \frac{2}{p^n}.$$

By comparison test the series converges if $p > 1$ and diverges if $0 < p \leq 1$.

4. We use the ratio test. If $a_N = n^n/n!$, then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{n^n(n+1)} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e > 1,$$

and so the series diverges.

5. We use root test. Let $a_n = (\sqrt[n]{n} - 1)^n$. Then

$$\sqrt[n]{a_n} = \sqrt[n]{n} - 1 \xrightarrow{n\to\infty} 0 < 1,$$

and so the series converges.

6. Let $a_n = (1+r^n)^{-1}$. If $|r| \leq 1$, then $\{a_n\}$ does not converge to zero, and so the series diverges. If $|r| > 1$, $\lim_{n\to\infty} r^{-n} = 0$, and so there exists an $N \in \mathbb{N}$ such that $1 + r^{-n} > 1/2$ for all $n > N$ (note that r could be negative, or else $1 + r^{-n}$ is of course bigger than 1). Then for $n > N$,

$$|a_n| = \frac{|r|^{-n}}{1 + r^{-n}} < \frac{2}{|r|^n},$$

and so by limit comparison test, the series is absolutely convergent for $|r| > 1$.

9. (a) Let $\{a_n\}$ be a sequence of of positive real numbers. Show that

$$\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

You may assume that each of the quantities is finite, even though the result holds true for extended reals. **Hint.** Proceed by contradiction. For instance, for the rightmost inequality, let $U = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and suppose $L > U$. Then use the equivalent characterizations of lim sup to draw a contradiction.

Solution: We will show that

$$\limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

The other inequalities also follow in a similar fashion. Denote $L = \limsup_{n\to\infty} \sqrt[n]{a_n}$ and $U = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$. We proceed by contradiction, so suppose $L > U$. Let $\beta \in (U, L)$. Then there exists an N such that for all $n > N$,

$$\left|\frac{a_{n+1}}{a_n}\right| < \beta,$$

or equivalently, $|a_{n+1}| < \beta|a_N|$. Inductively, one can conclude that $|a_n| < \beta^{n-N}|a_N|$. That is, for any $n > N$,

$$|a_n|^{1/n} < \beta^{1-N/n}|a_N|^{1/n}.$$

Now N is fixed, so taking limsup on both sides, since $\lim_{n\to\infty} \beta^{1-N/n}|a_N|^{1/n} = \beta$, we see that

$$\limsup_{n\to\infty} |a_n|^{1/n} \leq \beta,$$

which is a contradiction.

(b) Show that if $\sum a_n$ converges by the ratio test, then $\sum a_n$ also converges by the root test.

Solution: If $\sum a_n$ converges by the ratio test, then $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$. But then by the above set of inequalities, we necessarily have that $\limsup_{n\to\infty} \sqrt[n]{|a_n|} < 1$, and so the series also converges by the root test.

(c) Consider the sequence $\{a_n\}_{n=0}^{\infty}$,

$$a_n = \frac{1}{2^{n+(-1)^n}} = \begin{cases} \frac{1}{2^{n-1}}, & n \text{ is odd} \\ \frac{1}{2^{n+1}}, & n \text{ is even.} \end{cases}$$

Compute (with proper justifications) $\limsup \sqrt[n]{|a_n|}$ and $\limsup |a_{n+1}/a_n|$. Show that the series converges by the root test. Does the ration test work?

Solution: Note that

$$\sqrt[n]{|a_n|} = \frac{1}{2^{1+(-1)^n/n}} \xrightarrow{n\to\infty} \frac{1}{2},$$

and so $\limsup \sqrt[n]{|a_n|} = 1/2$. On the other hand,

$$\frac{a_{n+1}}{a_n} = \begin{cases} \frac{1}{8}, & n \text{ is odd} \\ 2, & n \text{ is even,} \end{cases}$$

and so $\limsup |a_{n+1}/a_n| = 2$ and $\liminf |a_{n+1}/a_n| = 1/8$. Since $\limsup \sqrt[n]{|a_n|} = 1/2 < 1$, the root test says that the series $\sum a_n$ converges. On the other hand since

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| < 1 < \limsup \left|\frac{a_{n+1}}{a_n}\right|,$$

the ratio test is inconclusive.

(d) Let $b_n = n^n/n!$. Show that

$$\lim_{n\to\infty} \sqrt[n]{b_n} = e.$$

Hint. It is easier to compute the limiting ratios.

Solution: Note that

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}n!}{(n+1)!n^n} = \frac{(n+1)^{n+1}}{(n+1)n^n} = \frac{(n+1)^n}{n^n} = \left(1+\frac{1}{n}\right)^n \xrightarrow{n\to\infty} e.$$

So $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = e$. But then by the chain of inequalities in the first part the middle two terms are also equal, that is, $\liminf_{n\to\infty} \sqrt[n]{a}_n = \limsup_{n\to\infty} \sqrt[n]{a}_n = e$, and so

$$\lim_{n\to\infty} \sqrt[n]{a_n} = e.$$

10. (a) Show that if $a_n > 0$, and $\lim_{n\to\infty} na_n = l \neq 0$, then $\sum a_n$ diverges.

Solution: Since $na_n \to l \neq 0$, applying the definition of convergence with $\varepsilon = |l|/2 > 0$, there exists $N \in \mathbb{N}$ such that

$$n > N \implies |na_n - l| < \frac{|l|}{2}.$$

In particular, for $n > N$, $a_n > |l|/2n$. By the comparison test, since $\sum 1/n$ diverges, it follows that $\sum a_n$ also diverges.

(b) Given that $\sum a_n$ converges *absolutely*, show that $\sum a_n^p$ also converges whenever $p > 1$. Give a counterexample, if $\sum a_n$ only converges conditionally.

> **Solution:** Since $\sum a_n$ converges absolutely, by the divergence test, $|a_n| \to 0$. In particular, there exists N such that for all $n > N$, $|a_n| < 1$. But then for any $p > 1$, $|a_n|^p < |a_n|$ when $n > N$. By comparison test, $\sum |a_n|^p$ converges, and hence $\sum a_n^p$ also converges. This is not true if $\sum a_n$ only converges conditionally. For instance, consider $a_n = (-1)^n/\sqrt{n}$ and $p = 2$.

11. Consider each of the following propositions. Provide short proofs for those that are true and counterexamples for any that are not.

(a) If $\sum a_n$ converges and the sequence $\{b_n\}$ also converges, then $\sum a_n b_n$ converges.

(b) If $\sum a_n$ converges conditionally, then $\sum n^2 a_n$ diverges.

> **Solution:** The proposition is true. If not, then $\sum n^2 a_n$ converges, and so $\lim_{n\to\infty} n^2 a_n = 0$. In particular, there exists N such that for all $n > N$, $|a_n| < 1/n^2$, and so $\sum a_n$ must converge absolutely by the comparison test. A contradiction!

(c) If $\{a_n\}$ is a decreasing sequence, and $\sum a_n$ converges, then $\lim_{n\to\infty} n a_n = 0$.

> **Solution:** The proposition is true. Since $\sum a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$. But then since a_n is also decreasing, it follows that $a_n \geq 0$. By the Cauchy criteria, given any $\varepsilon > 0$, there exists an N such that for all $n > m > N$,
>
> $$\sum_{k=m}^{n} a_k < \frac{\varepsilon}{2}.$$
>
> Applying this to $m = \lfloor n/2 \rfloor$ with $n > 2N$, and using the fact that a_n decreases
>
> $$\frac{\varepsilon}{2} > \sum_{k=\lfloor n/2 \rfloor}^{n} a_k > \frac{n a_n}{2}.$$
>
> So given $\varepsilon > 0$, if $n > 2N$, then $|n a_n| < \varepsilon$, and hence $\lim_{n\to\infty} n a_n = 0$.

12. (a) For any $n \in \mathbb{N}$, show that the function $p_n(x) = x^n$ is continuous on all of $\mathbb{R}$. Show the explicit dependence of δ on ε and the point that you are looking at.

> **Solution:** We prove continuity at $x = a$. Let $\varepsilon > 0$ be given. We need to estimate
>
> $$\begin{aligned} |p_n(x) - p_n(a)| &= |x^n - a^n| \\ &= |x - a||x^{n-1} + x^{n-2}a + \cdots + a^{n-1}|. \end{aligned}$$
>
> Now, $|x - a| < \delta$, where $\delta > 0$ is to be chosen. Suppose, we choose $\delta < 1$, then clearly $|x| < |a| + 1$. A general term on the right is of the form $x^j a^{n-1-j}$ for $j = 0, \cdots, n-1$. So if $\delta < 1$, we have
>
> $$|x^j a^{n-1-j}| < (|a| + 1)^j |a|^{n-1-j} < (|a| + 1)^{n-1}.$$
>
> Then by triangle inequality,
>
> $$|x^{n-1} + x^{n-2}a + \cdots + a^{n-1}| < n(1 + |a|)^{n-1}.$$

So if $\delta < 1$ and $|x-a| < \delta$, then

$$|p_n(x) - p_n(a)| < n\delta(1+|a|)^{n-1}.$$

Our aim is to make this smaller than ε, and so simply choose

$$\delta < \frac{\varepsilon}{n(1+|a|)^{n-1}}.$$

Together with $\delta < 1$, we see that if

$$\delta = \min\left(1, \frac{\varepsilon}{n(1+|a|)^{n-1}}\right),$$

then

$$|x-a| < \delta \implies |p_n(x) - p_n(a)| < \varepsilon.$$

(b) Show that $f(x) = \sqrt{x}$ is continuous on $(0, \infty)$.

Solution: Let $\varepsilon > 0$, and $|x-a| < \delta$ for some $\delta > 0$ to be chosen later. Clearly

$$|\sqrt{x} - \sqrt{a}| = \frac{|x-a|}{|\sqrt{x}+\sqrt{a}|} < \frac{\delta}{|\sqrt{x}+\sqrt{a}|}.$$

Now, $a > 0$, by choosing $\delta < a/2$,

$$|x-a| < \delta \implies x > a/2.$$

So

$$\sqrt{x} + \sqrt{a} > 3\sqrt{a}/2,$$

and

$$|\sqrt{x} - \sqrt{a}| < \frac{2\delta}{3\sqrt{a}}.$$

So simply pick

$$\delta = \min\left(\frac{a}{2}, \frac{3\sqrt{a}}{2}\varepsilon\right).$$

(c) Show that $f_n(x) = x^{1/n}$ is continuous on $(0, \infty)$.

Solution: Again, we prove continuity at $x = a$. Let $\varepsilon > 0$. From the identity, we see that

$$x - a = (x^{1/n} - a^{1/n})(x^{1-1/n} + x^{1-2/n}a^{1/n} \cdots + a^{1-1/n})$$

Then

$$|f_n(x) - f_n(a)| = \frac{|x-a|}{|x^{1-1/n} + x^{1-2/n}a^{1/n} \cdots + a^{1-1/n}|}.$$

Again as before, since $a > 0$, if $\delta < a/2$, then $x > a/2$ and so

$$x^{1-1/n} + x^{1-2/n}a^{1/n} \cdots + a^{1-1/n} > na^{1-1/n}c_n,$$

where c_n is the constant (independent of a)

$$c_n = 2^{1-1/n} + 2^{1-2/n} + \cdots + 1.$$

And so if $|x-a|<\delta$ and $\delta<a/2$ we have

$$|f_n(x)-f_n(a)|<\frac{\delta}{nc_n a^{1-1/n}}.$$

So simply pick

$$\delta=\min\left(\frac{a}{2},\ nc_n a^{1-1/n}\varepsilon\right).$$

Hint. For all parts the following identity might be useful.

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1}).$$