Department of Mathematics, University of California, Berkeley

*STUDENT EXAM NUMBER* ________________

GRADUATE PRELIMINARY EXAMINATION, Part A — Spring Semester 2013

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1. Please write your 1- or 2-digit student exam number on this cover sheet and on **all** problem sheets (even problems that you do not wish to be graded).
2. Indicate below which six problems you wish to have graded. **Cross out** solutions you may have begun for the problems that you have not selected.
3. Extra sheets should be stapled to the appropriate problem at the upper right corner. Do not put work for problem $p$ on either side of the page for problem $q$ if $p \neq q$.
4. No notes, books, or calculators may be used during the exam.

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PROBLEM SELECTION

Part A: List the six problems you have chosen:

______ , ______ , ______ , ______ , ______ , ______

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GRADE COMPUTATION

| | | |
|---|---|---|
| 1A. ______ | 1B. ______ | Calculus |
| 2A. ______ | 2B. ______ | Real analysis |
| 3A. ______ | 3B. ______ | Real analysis |
| 4A. ______ | 4B. ______ | Complex analysis |
| 5A. ______ | 5B. ______ | Complex analysis |
| 6A. ______ | 6B. ______ | Linear algebra |
| 7A. ______ | 7B. ______ | Linear algebra |
| 8A. ______ | 8B. ______ | Abstract algebra |
| 9A. ______ | 9B. ______ | Abstract algebra |

Part A Subtotal: ______ Part B Subtotal: ______ Grand Total: ______

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 1A.** *Score:*

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Suppose $f : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function. Calculate the limit

$$\lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} f(t) \frac{\epsilon}{\epsilon^2 + t^2} dt$$

**Solution:**

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*Please cross out this problem if you do not wish it graded*

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**Problem 2A.** *Score:*

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Suppose that $f$ is a smooth real function defined for all real $x$, such that $|f'(x)| \geq \epsilon > 0$ and $|f''(x)| \leq M > 0$ for all $x$.

(1) Show that $f$ has a unique zero $z$.

(2) Given $x_0$, define a sequence by $x_{n+1} = x_n - f(x_n)/f'(x_n)$. Show that

$$|x_{n+1} - z| \leq |x_n - z|^2 M/\epsilon.$$

(Hint: $f(x_n) = \int_z^{x_n} f'(x)dx$.)

(3) Show that the sequence $\{x_n\}$ converges to the zero $z$ of $f$ provided that $|f(x_0)| < \epsilon^2/M$.

**Solution:**

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 3A.** *Score:*

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Show that $\int_0^\infty x \exp(-x^6 (\sin x)^2) dx$ is finite.

**Solution:**

*STUDENT EXAM NUMBER* ______________________

*Please cross out this problem if you do not wish it graded*

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**Problem 4A.** *Score:*

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Find

$$\int_C \frac{\cosh(\pi z)}{z(z^2+1)}\, dz$$

when $C$ is the circle $|z| = 2$, described in the positive sense.

**Solution:**

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 5A.** *Score:*

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Let $n \geq 1$ and let $\{a_0, a_1, \ldots, a_n\}$ be complex numbers such that $a_n \neq 0$. For $\theta \in \mathbf{R}$, define

$$f(\theta) = a_0 + a_1 e^{i\theta} + a_2 e^{2i\theta} + \ldots + a_n e^{ni\theta}.$$

Prove that there exists $\theta \in \mathbf{R}$ such that $|f(\theta)| > |a_0|$.

**Solution:**

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 6A.** *Score:*

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Show that if $V$ is a real vector space with a positive definite symmetric bilinear form $\langle \cdot, \cdot \rangle$ and $W \subset V$ is a linear subspace then $W^{\perp} = ((W^{\perp})^{\perp})^{\perp}$. Give an example such that $W \neq (W^{\perp})^{\perp}$.

**Solution:**

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 7A.** *Score:*

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Let $A$ be a matrix over the field of complex numbers. Suppose $A$ has finite order, in other words $A^m = I$ for some positive integer $m$. Prove that $A$ is diagonalizable. Give an example of a matrix of finite order over an algebraically closed field that is not diagonalizable.

**Solution:**

*STUDENT EXAM NUMBER* ______________________

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**Problem 8A.** *Score:*

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Let $m$ and $n$ be integers greater than 1. Prove that $\log_m(n)$ is rational if and only if $m = l^r$ and $n = l^s$, for some positive integers $l$, $r$, and $s$.

**Solution:**

*STUDENT EXAM NUMBER* ____________________

*Please cross out this problem if you do not wish it graded*

**Problem 9A.** *Score:*

Let $K$ be a field. Let $R$ be an integral domain which contains $K$ and is finite-dimensional (as a vector space) over $K$. Prove that $R$ is a field.

**Solution:**

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GRADUATE PRELIMINARY EXAMINATION, Part B Spring Semester 2013

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PROBLEM SELECTION

Part B: List the six problems you have chosen:

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*Please cross out this problem if you do not wish it graded*

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**Problem 1B.** *Score:*

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Find $\int_0^1 \arctan(x)dx$.

**Solution:**

*STUDENT EXAM NUMBER* ______________________

*Please cross out this problem if you do not wish it graded*

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**Problem 2B.** *Score:*

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Prove that the intersection of a decreasing sequence of closed connected subsets of a compact metric space is connected. Give an example to show that this is false if the assumption that the space is compact is dropped.

**Solution:**

*STUDENT EXAM NUMBER* ______________________

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**Problem 3B.** *Score:*

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Let $g$ be $2\pi$-periodic, continuous on $[-\pi, \pi]$ and have Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \, .$$

Let $f$ be $2\pi$-periodic and satisfy the differential equation

$$f''(x) + kf(x) = g(x)$$

where $k \neq n^2, n = 1, 2, 3, \ldots$. Find the Fourier series of $f$ and prove that it converges everywhere.

**Solution:**

*STUDENT EXAM NUMBER* ____________________

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**Problem 4B.** *Score:*

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Let $U$ be an open subset of $\mathbf{C}$. Let $K$ be a closed bounded subset of $\mathbf{C}$ that is contained in $U$. Put

$$D = \min_{p \in K, q \notin U} |p - q|.$$

That is, $D$ is the closest distance between $K$ and $\mathbf{C} - U$. (If $U = \mathbf{C}$ then we put $D = \infty$.)

Suppose that $f$ is an analytic function on $U$ so that for all $z \in U$, we have $|f(z)| \leq M$. Here $M$ is a fixed positive number. Find an explicit number $C < \infty$, depending on $M$ and $D$, so that for all $z_0 \in K$ we have $|f'(z_0)| \leq C$. Justify your answer.

**Solution:**

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*Please cross out this problem if you do not wish it graded*

**Problem 5B.** *Score:*

Which of the following domains are biholomorphically equivalent to each other: the complex plane $\mathbb{C}$, the unit disk $D \subset \mathbb{C}$, the upper halfplane $\mathbb{H} \subset \mathbb{C}$? Write explicit biholomorphisms or prove they cannot exist.

**Solution:**

*STUDENT EXAM NUMBER* ________________________

*Please cross out this problem if you do not wish it graded*

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**Problem 6B.** *Score:*

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Show that the $n \times n$ (Cauchy) matrix with entries $1/(x_i - y_j)$ has determinant

$$\frac{\prod_{1 \le j < i \le n}(x_i - x_j)(y_j - y_i)}{\prod_{1 \le i,j \le n}(x_i - y_j)}$$

**Solution:**

*STUDENT EXAM NUMBER* ______________________

*Please cross out this problem if you do not wish it graded*

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**Problem 7B.** *Score:*

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Prove the following three statements about real $n \times n$ matrices.

1. If $A$ is an orthogonal matrix whose eigenvalues are all different from $-1$, then $I + A$ is nonsingular and $S = (I - A)(I + A)^{-1}$ is skew-symmetric.
2. If $S$ is a skew-symmetric matrix, then $A = (I - S)(I + S)^{-1}$ is an orthogonal matrix with no eigenvalue equal to $-1$.
3. The correspondence (called the Cayley transform) $A \leftrightarrow S$ from Parts 1 and 2 is one-to-one.

**Solution:**

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**Problem 8B.** *Score:*

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Consider the symmetric group $\Sigma_n$ in its presentation as $n \times n$ permutation matrices. Define the "expected trace" to be the weighted sum of traces

$$E_n = \frac{1}{n!} \sum_{g \in \Sigma_n} \mathrm{Trace}(g)$$

Calculate $E_n$.

**Solution:**

*STUDENT EXAM NUMBER* ______________________
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**Problem 9B.** *Score:*

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If $F$ is a finite field, show that more than half the elements of $F$ are squares. Show that every element is the sum of 2 squares.

**Solution:**