Problem 1A. *Score:*

Show that

$$\int_4^9 \sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{-6+5\sqrt{x}}}}}\,dx$$

is a rational number.

Solution: Let $f(x)$ be the function in the integrand. Note that $f(x)$ is defined on $[4,9]$ as $4 \le -6+5\sqrt{x} \le 9$ whenever $4 \le x \le 9$. Also, $f(x)$ is strictly increasing and $f(4)=2$ and $f(9)=3$. So $f\colon [4,9] \to [2,3]$ is invertible. Its inverse is

$$f^{-1}(y) = \left(\frac{\left(\frac{\left(\frac{\left(\frac{y^2+6}{5}\right)^2+6}{5}\right)^2+6}{5}\right)^2+6}{5}\right)^2$$

which is a polynomial with rational coefficients.

The integral $\int_4^9 f(x)dx$ is equal to the area of the region bounded by the graph $y=f(x)$, the vertical lines $x=4, x=9$ and the x-axis. The union of this region with the region bounded by the graph $y=f(x)$, the horizontal lines $y=2, y=3$ and the y-axis is the difference between two rectangles: one bounded by the lines $x=9, y=3$ and the x,y-axes and the other bounded by the lines $x=4, y=2$ and the x,y-axes. Thus

$$\int_4^9 f(x)dx + \int_2^3 f^{-1}(y)dy = 9\cdot 3 - 4\cdot 2.$$

The second integral is a rational number, since $f^{-1}(y)$ is a polynomial with rational coefficients. So the first integral is also a rational number.

Problem 2A. *Score:*

Suppose that f and g are continuously differentiable real-valued functions on $\mathbb{R}$ with $f,g,f',g' \in L^2(\mathbb{R})$. Show that

$$\int_{-\infty}^{\infty} fg'\,dx = -\int_{-\infty}^{\infty} f'g\,dx.$$

(Recall that $L^2(\mathbb{R})$ is the set of integrable functions h such that $\int_{-\infty}^{\infty} |h|^2\,dx < \infty$.)

Solution: The condition $f,g,f',g' \in L^2(\mathbb{R})$ implies that the two integrals exist. It also implies that $\int_{-\infty}^{\infty} |f(x)|\cdot|g(x)|\,dx < \infty$, and hence that there are sequences $\{x_i\}_{i=1}^{\infty}$ and $\{y_i\}_{i=1}^{\infty}$

such that $\lim_{i\to\infty} x_i = -\infty$, $\lim_{i\to\infty} y_i = \infty$ and $\lim_{i\to\infty} f(x_i)g(x_i) = \lim_{i\to\infty} f(y_i)g(y_i) = 0$. Then

$$\int_{-\infty}^{\infty} (fg' + f'g)\, dx \;=\; \lim_{i\to\infty} \int_{x_i}^{y_i} (fg' + f'g)\, dx \;=\; \lim_{i\to\infty} (f(y_i)g(y_i) - f(x_i)g(x_i)) = 0.$$

Problem 3A. *Score:*

Suppose g and f_n are nonnegative integrable functions such that $\int f_n\, dx \to 0$ as $n \to \infty$ and $f_n^2 \leq g$ for all n. Prove or find a counterexample to the statement that $\int f_n^4\, dx \to 0$ as $n \to \infty$.

Solution: For a counterexample we can take the domain of the functions to be $(0, 1)$, take f_n to be the step function

$$f_n(x) = \begin{cases} n^{1/4} & 0 < x \leq 1/n \\ 0 & 1/n < x < 1, \end{cases}$$

and take $g(x) = x^{-1/2}$. The condition $f_n^2 \leq g$ is satisfied, g is integrable with $\int g\, dx = 2$, f_n is integrable with $\int f_n\, dx = n^{-3/4} \to 0$, and $\int f_n^4\, dx = 1$ for all n.

Problem 4A. *Score:*

Prove that a monic polynomial $p(z)$ with real coefficients is real-rooted if and only if $\Im(p'(z)/p(z)) < 0$ whenever $\Im(z) > 0$. ($\Im(z)$ denotes the imaginary part of z.)

Solution: ($\Rightarrow$) Let $p(z) = \prod_{i=1}^{n} (z - \lambda_i)$ and observe that

$$\frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - \lambda_i} = \sum_{i=1}^{n} \frac{\overline{z} - \overline{\lambda_i}}{|z - \lambda_i|^2}.$$

Since the λ_i are real all the numerators are in the lower half plane for z in the upper half plane, so any linear combination of them with nonnegative coefficients is also in the lower half plane.

($\Leftarrow$) If p is not real-rooted then it must have a zero in the upper half plane, since the zeros occur in conjugate pairs. Let λ be such a zero, occuring with multiplicity m, and observe that

$$\frac{p'}{p}(\lambda - \epsilon i) = \frac{m}{-\epsilon i} + \frac{q'}{q}(\lambda - \epsilon i),$$

where $q(z) = p(z)/(z-\lambda)^m$. Since $q(\lambda) \neq 0$ we find that $q'(z)/q(z)$ is bounded in a neighborhood of λ, so

$$\lim_{\epsilon \to 0} \frac{m}{-\epsilon i} + \frac{q'}{q}(\lambda - \epsilon i) = i\infty,$$

in particular yielding a z for which $\Im(z) > 0$ and $\Im(p'(z)/p(z)) > 0$.

Problem 5A. *Score:*

Compute

$$\int_0^{2\pi} \frac{d\theta}{(3+e^{-i\theta})^2}.$$

Solution: Put $z = e^{i\theta}$. Then

$$\int_0^{2\pi} \frac{d\theta}{(3+e^{-i\theta})^2} = \int_{|z|=1} \frac{1}{(3+z^{-1})^2}\frac{dz}{iz} = 2\pi \operatorname{Res}_{z=-\frac{1}{3}} \frac{z}{(3z+1)^2} = \frac{2\pi}{9}.$$

Problem 6A. *Score:*

Prove or disprove: there exists an $\epsilon > 0$ and a real matrix A such that

$$A^{100} = \begin{bmatrix} -1 & 0 \\ 0 & -1-\epsilon \end{bmatrix}.$$

Solution: The eigenvalues $a, b \in \mathbb{C}$ of such a matrix A must satisfy

$$a^{100} = -1 \quad b^{100} = (-1-\epsilon).$$

Note that a cannot be real since 100 is even. Moreover, since A is real its characteristic polynomial is real so we must have $a = \bar{b}$. But this is impossible since $|a|^{100} = 1$ and $|b|^{100} = (1+\epsilon)^{100} \neq 1$. So no such matrix can exist.

Problem 7A. *Score:*

Suppose A is a symmetric matrix with rational entries and $A = UDU^T$, where U is orthogonal. Must D have rational entries? Prove or find a counterexample.

Solution: Since U is orthogonal the factorization $A = UDU^T$ diagonalizes A so the entries of D are the eigenvalues of A. These are the roots of its characteristic polynomial, which need not be rational. For instance, consider

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$(x-1)(x+1) - 1 = x^2 - 2,$$

which has roots $\pm\sqrt{2}$.

Problem 8A. *Score:*

Find a product of cyclic groups of prime power order isomorphic to the group of units in the ring of integers modulo 2016.

Solution: $2016 = 2^5 \times 3^2 \times 7$, so the group of units is the product of the groups of units of the integers mod 2^5, 3^2, 7, which are products of cyclic groups of orders 2, 8 and 2, 3 and 2, 3. So the solution is that the group is a product of cyclic groups of orders 2, 2, 2, 3, 3, 8.

Problem 9A. *Score:*

Compute the Galois group of the normal closure of the field

$$K = \mathbb{Q}(\sqrt{3} + \sqrt{5})$$

over $\mathbb{Q}$.

Solution: We first prove $K = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. For that, it suffices to prove $\sqrt{3}, \sqrt{5} \in K$. In fact, by

$$(\sqrt{3} + \sqrt{5})^2 = 8 + 2\sqrt{15},$$

we have $\sqrt{15} \in K$. Then

$$(\sqrt{3} + \sqrt{5})\sqrt{15} = 5\sqrt{3} + 3\sqrt{5}$$

is also in K. Its $\mathbb{Q}$-linear combinations with $\sqrt{3} + \sqrt{5}$ give $\sqrt{3}, \sqrt{5} \in K$.

As a consequence, K is Galois over $\mathbb{Q}$, since it is the composite of two Galois extensions over $\mathbb{Q}$. The normal closure of K over $\mathbb{Q}$ is still K.

We next prove $\mathbb{Q}(\sqrt{3}) \cap \mathbb{Q}(\sqrt{5}) = \mathbb{Q}$. Otherwise, we would have $\mathbb{Q}(\sqrt{3}) = \mathbb{Q}(\sqrt{5})$ since both of them have degree 2 over $\mathbb{Q}$. As a consequence, we have $\sqrt{5} = a\sqrt{3} + b$ for some

$a, b \in \mathbb{Q}$. Taking squares, we have $5 = 3a^2 + b^2 + 2ab\sqrt{3}$. We must have $ab = 0$, so $a = 0$ or $b = 0$. If $a = 0$, $\sqrt{5} = b \in \mathbb{Q}$ is a contradiction. If $b = 0$, $\sqrt{5/3} = a \in \mathbb{Q}$ is still a contradiction.

Finally, the composite gives

$$\mathrm{Gal}(K/\mathbb{Q}) = \mathrm{Gal}(\mathbb{Q}(\sqrt{3})/\mathbb{Q}) \times \mathrm{Gal}(\mathbb{Q}(\sqrt{5})/\mathbb{Q}) \simeq (\mathbb{Z}/2\mathbb{Z})^2.$$

An alternate approach is to start by by showing, as above, that $\sqrt{5} \notin \mathbb{Q}(\sqrt{3})$, and define $L = \mathbb{Q}(\sqrt{3}, \sqrt{5})$. Then L has degree 4 over $\mathbb{Q}$ and $\mathrm{Aut}_{\mathbb{Q}}(L)$ contains a group G isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, generated by $\sigma\colon \sqrt{3} \leftrightarrow -\sqrt{3}$ and $\tau\colon \sqrt{5} \leftrightarrow -\sqrt{5}$. It follows that L is Galois over $\mathbb{Q}$ with Galois group G. Since the stabilizer of $\sqrt{3} + \sqrt{5}$ in G is trivial, $K = L$.

Problem 1B. *Score:*

Show that

$$\int_0^\infty \frac{t\,e^{-t/2}}{1 - e^{-t}}\,dt = 4\sum_{n=0}^\infty \frac{1}{(2n+1)^2}$$

Solution:

Expand $e^{-t/2}/(1 - e^{-t})$ in a geometric series and integrate

$$\int_0^\infty \sum_{n=0}^\infty t\,e^{-(n+1/2)t}\,dt.$$

term by term, using the formula

$$\int_0^\infty e^{-\alpha t}\,t\,dt = \frac{1}{\alpha^2},$$

valid for any $\alpha > 0$. This formula can be obtained using integration by parts.

(It is also possible, but more difficult, to evaluate both sides of the required identity explicitly, obtaining the value $\pi^2/2$.)

Since this is meant to be a calculus problem rather than a real analysis problem, graders should not demand justification for the interchange of summation and integration.

Problem 2B. *Score:*

Let $(f_i)_{i=1}^\infty$ and g be twice-differentiable real-valued functions on $\mathbb{R}$, with $f_i'' \geq 0$. Suppose that

$$\lim_{i\to\infty} f_i(x) = g(x)$$

for all $x \in \mathbb{R}$. Show that $g'' \geq 0$.

Solution: Since each f_i is concave upward, we have

$$f_i(z) \leq \frac{f_i(z+h) + f_i(z-h)}{2}$$

for all $z, h \in \mathbb{R}$. It follows that g satisfies the same inequality. Then

$$g''(z) = \lim_{h \to 0} \frac{g(z+h) + g(z-h) - 2g(z)}{h^2} \geq 0$$

for all $z \in \mathbb{R}$.

Problem 3B. *Score:*

Show that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k + |x|}$$

converges pointwise to a Lipschitz function $f(x)$. Is the convergence uniform on $\mathbb{R}$?

Solution: Since the series is alternating for every x, it converges pointwise to some limiting function $f(x)$. For the Lipschitz condition, we can assume without loss of generality that $x \geq 0$ and compute

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k+x} - \sum_{k=1}^{\infty} \frac{(-1)^k}{k+x+\epsilon} = \sum_{k=1}^{\infty} \frac{(-1)^k \epsilon}{(k+x)(k+x+\epsilon)}.$$

The last sum is again an alternating series, bounded in absolute value by its first term $\epsilon/((1+x)(1+x+\epsilon)) \leq \epsilon$. Hence $f(x)$ is Lipschitz with constant 1.

Using again the fact the series is alternating, its N^{th} tail has

$$\left| f(x) - \sum_{k=1}^{N-1} \frac{(-1)^k}{k+|x|} \right| = \left| \sum_{k=N}^{\infty} \frac{(-1)^k}{k+|x|} \right| \leq \frac{1}{N+|x|} \leq \frac{1}{N},$$

so it converges uniformly.

Problem 4B. *Score:*

Compute

$$\int_C \frac{6z^5+1}{z^6+z+1}\, dz,$$

where C is the circle centered at the origin with radius 2, oriented counterclockwise.

Solution: If $|z| > 3/2$ then $|z+1| \le |z|+1 < |z|^6 = |z^6|$, so $\frac{6z^5+1}{z^6+z+1}$ is analytic on $\{z\colon |z| > 3/2\}$. Then

$$\int_C \frac{6z^5+1}{z^6+z+1}\,dz = -2\pi i \ \mathrm{Res}_{z=\infty} \frac{6z^5+1}{z^6+z+1} = 12\pi i.$$

Alternate solution: the integrand is the derivative of $\log(z^6+z+1)$. Hence the integral is i times the change in $\arg(z^6+z+1)$ along C, or $2\pi i$ times the number of zeroes of z^6+z+1 inside C. The same argument as above shows that all six zeroes are inside C.

Problem 5B. *Score:*

Let $f(z) = \sum f_n z^n$ and $g(z) = \sum g_n z^n$ define holomorphic functions on a neighborhood of the closed unit disk $D = \{z : |z| \le 1\}$. Prove that $h(z) = \sum f_n g_n z^n$ also defines a holomorphic function on a neighborhood of D.

Solution: The series for $f(z)$ and $g(z)$ must have radius of convergence greater than 1. Hence there is a $\rho > 1$ such that $f_n/\rho^n \to 0$ and $g_n/\rho^n \to 0$ as $n \to \infty$. Then $f_n g_n/\rho^{2n} \to 0$ implies that the series for $h(z)$ has radius of convergence at least ρ^2, which is again greater than 1.

Problem 6B. *Score:*

Let A be an $m \times n$ real matrix and $y \in \mathbb{R}^m$. Let $x \in \mathbb{R}^n$ be a vector with nonnegative entries that minimizes the Euclidean distance $\|y - Ax\|$ (among all nonnegative vectors x). Show that the vector $v = A^T(y - Ax)$ has nonnegative entries.

Solution: Suppose $v_j = a_j^T(Ax - y) < 0$, where $a_j = Ae_j$ is the j-th column of A. Then for sufficiently small $\epsilon > 0$,

$$\|y - A(x + \epsilon e_j)\|^2 = \|y - Ax - \epsilon a_j\|^2 = \|y - Ax\|^2 + 2\epsilon a_j^T(Ax - y) + \epsilon^2 \|a_j\|^2 < \|y - Ax\|^2,$$

contrary to the hypothesis on x.

Problem 7B. *Score:*

Let A be a real square matrix and let ρ be the maximum of the absolute values of its eigenvalues (*i.e.*, its spectral radius). (1) Show that if A is symmetric then $\|Ax\| \leq \rho\|x\|$ for all $x \in \mathbb{R}^n$, where $\|\cdot\|$ denotes the Euclidean norm. (2) Is this true when A is not symmetric? Prove or give a counterexample.

Solution: (1) Assume A is $n \times n$. Since A is symmetric it has real eigenvalues $\lambda_1 \ldots \lambda_n$ and orthogonal eigenvectors $u_1, \ldots, u_n$. Thus, by the spectral theorem:

$$A^2 = \sum_{i=1}^{n} \lambda_i^2 u_i u_i^T.$$

This implies that for any x:

$$\|Ax\|^2 = x^T A^2 x = \sum_{i=1}^{n} \lambda_i^2 \langle x, u_i \rangle^2 \leq \left(\max_i \lambda_i^2\right) \|x\|^2,$$

since $\sum_{i=1}^{n} \langle x, u_i \rangle^2 = \|x\|^2$. Taking square roots proves the claim.

(2) This is not true. Consider the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

which has both eigenvalues equal to zero, but

$$Ae_2 = e_1,$$

for elementary basis vectors e_1, e_2.

Problem 8B. *Score:*

Factor the polynomial

$$f(x) = 6x^5 + 3x^4 - 9x^3 + 15x^2 - 13x - 2$$

into a product of irreducible polynomials in the ring $\mathbb{Q}[x]$.

Solution: Since $f(1) = 0$, the polynomial has a factor $x - 1$. Then we obtain

$$f(x) = (x - 1)(6x^4 + 9x^3 + 15x + 2).$$

We claim that this is a final form of the factorization, for which we need to prove that

$$g(x) = 6x^4 + 9x^3 + 15x + 2$$

is irreducible.

First, the polynomial

$$h(x) = 2x^4 + 15x^3 + 9x + 6$$

is irreducible. This follows from the Eisenstein criterion by the prime number $p = 3$.

Second, for any polynomial

$$\phi(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$$

of degree n, denote

$$\tilde{\phi}(x) = x^n \phi(x^{-1}) = a_0 x^n + \cdots + a_{n-1} x + a_n.$$

Then $g(x) = \tilde{h}(x)$, and a decomposition $g(x) = g_1(x) g_2(x)$ would give a decomposition

$$h(x) = \tilde{g}_1(x) \tilde{g}_2(x).$$

Hence, g is irreducible.

Problem 9B. *Score:*

Let p be a prime number. Prove that every group G of order p^2 is commutative.

Solution: Let G act on itself by conjugation, $g(x) = gxg^{-1}$. Under the action, G is a disjoint union of orbits O_0, O_1, $\ldots$, O_r, where $O_0 = \{e\}$ is the orbit of the identity element. The length $|O_i|$ of each orbit is a divisor of $|G| = p^2$, so is equal to 1, p, or p^2. We have the sum

$$|O_0| + |O_1| + \cdots + |O_r| = |G| = p^2.$$

Since $|O_0| = 1$, at least $p - 1$ other orbits O_i must have length 1. Let $O_{i_0} = \{x_0\}$ be such an orbit. By definition, $gx_0g^{-1} = x_0$ for all $g \in G$. Then x_0 is in the center of G, and it is not the identity.

If x_0 generates G, then $G \cong \mathbb{Z}/p^2\mathbb{Z}$ is commutative. If x_0 does not generate G, then it generates a subgroup $\langle x_0 \rangle$ of order p. Let x_1 be any element of $G - \langle x_0 \rangle$. Then the subgroup $\langle x_0, x_1 \rangle$ has order greater than p, thus it is equal to G. Since x_0 and x_1 commute, we see that G is still commutative.