MATH 8150 Final Exam — Spring 2020

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Print Your Name: ______________________________

You may freely use the theorems we covered in the semester
(either proved in the text or in class).

You may use results in homework assigned during the semester *unless*
they are not what the exam problems explicitly ask you to do and you state them clearly.

If you have work on separate pages for grading, clearly label each problem.
Cross out the parts you do not want to be graded.

Problem #	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	120	

!!! GOOD LUCK AND HAVE FUN!!!

1. Compute the integral $\int_0^{2\pi} \frac{\sin^2\theta}{a+b\cos\theta} d\theta$ with $a,\ b \in \mathbb{R},\ |a| > |b| > 0$.

2. The function $f(z) = \frac{1}{(z-1)z^2}$ is analytic in $\mathbb{C} \setminus \{0, 1\}$.

   (i) Expand this function in a Laurent series valid in a deleted neighborhood of (a) $z = 0$ and (b) $z = 1$.

   (ii) Expand $f(z)$ in Laurent series centered at $z = 0$ that are valid in (a) $|z| < 1$, (b) $1 < |z| < 2$, (c) $z| > 2$

3. (1) Show that if f is analytic in an open set containing the disc $|z - a| \leq R$, then

$$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r dr d\theta$$

   (2) Let Ω be a region and $M > 0$ a fixed positive constant. Let $\mathcal{F}$ be the family of all analytic functions f on Ω such that $\iint_\Omega |f(z)|^2 dxdy \leq M$. Show that $\mathcal{F}$ is a normal family.

4. A holomorphic mapping $f : U \to V$ is a local bijection on U if for every $z \in U$ there exists an open disc $D \subset U$ centered at z so that $f : D \to f(D)$ is a bijection. Prove that a holomorphic map $f : U \to V$ is a local bijection if and only if $f'(z) \neq 0$ for all $z \in U$.

5. Consider the function $f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ for $z \in \mathbb{C} \setminus \{0\}$.

   (a) Show that f is univalent on the punctured disc $\mathbb{D} \setminus \{0\}$. What is the image of $|z| = r < 1$ under this map?

   (b) Show that f is univalent on the domain $\{z \in \mathbb{C} : \ |z| > 1\}$. What is the image of this domain under this map?

   (c) What is the inverse mapping $f^{-1} : \mathbb{C} \setminus [-1, 1] \to \mathbb{D} \setminus \{0\}$?

6. (a) (The maximum modulus principle) Suppose that U is a bounded domain and that $f(z)$ is a non-constant continuous function on $\bar{U}$ whose restriction to U is holomorphic. If $z_0 \in U$ that

$$|f(z_0)| < \sup\{|f(z)| : \ z \in \partial U\}.$$

   (b) Furthermore if $|f(z)|$ is constant on ∂U, then $f(z)$ has a zero in U: there exists $z_0 \in U$ for which $f(z_0) = 0$.