

MATH 8150 Midterm Exam — Spring 2020

Instructor: Prof. Jingzhi Tie

Please read the UGA Student Honor Code: “I will be academically honest in all of my academic work and will not tolerate academic dishonesty of others.”

SIGNATURE: _____

Print Your Name: _____

Due date: April 20, 2020

You can freely use the theorems we have covered in the semester (either proved in the text or in class). Clearly label each problem in your paper. Cross out the parts you do not want to be graded.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
6	20	
Total	120	

1. Prove that the distinct complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

2. Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be analytic and one-to-one in $|z| < 1$. For $0 < r_0 < 1$, let \overline{D}_{r_0} be the closed disk $|z| \leq r_0$. Show that the area A of $f(\overline{D}_{r_0})$ is finite and is given by

$$A = \pi \sum_{n=1}^{\infty} n |c_n|^2 r_0^{2n}.$$

[Hint: First find a formula in terms of polar coordinates in xy -plane for the area element $dudv$ using complex analysis, where $f = u + iv$. Note that $dxdy = rdrd\theta$.]

3. Assume f is continuous in the region: $R_0 \leq |z - a| < \infty$, $0 \leq \arg(z - a) \leq \beta_0$ ($0 < \beta_0 \leq 2\pi$) and the limit $\lim_{z \rightarrow \infty} (z - a)f(z) = A$ exists. Show that

$$\lim_{r \rightarrow +\infty} \int_{\gamma_r} f(z) dz = iA\beta_0 ,$$

where $\gamma_r := \{z \mid z = a + re^{it}, 0 \leq t \leq \beta_0\}$.

4. Computer the integral $I(b) = \int_0^{\frac{\pi}{2}} (\tan t)^{ib} dt$ for $b \in \mathbb{R}$. Hint: Some simple substitution will reduce the integral to what we have done in homework and lectures.
5. Let $f(z) = 2z^5 + 8z - 1$. Show that all five zeros of $f(z)$ are inside the disk $D(0, 2)$ and only one zero is inside the disk $D(0, 1)$.
6. (Cauchy's formula for "exterior" region) Let γ be piecewise smooth simple closed curve with interior Ω_1 and exterior Ω_2 . Assume $f'(z)$ exists in an open set containing γ and Ω_2 and $\lim_{z \rightarrow \infty} f(z) = A$. Show that

$$\frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)}{\xi - z} d\xi = \begin{cases} A, & \text{if } z \in \Omega_1, \\ -f(z) + A, & \text{if } z \in \Omega_2 \end{cases}$$