Preliminary Exam - Summer 1979

Problem 1 *Prove that the matrix*

$$\begin{pmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{pmatrix}$$

has two positive and two negative eigenvalues (counting multiplicities).

Problem 2 *Let* $\mathbf{F}$ *be a subfield of a field* $\mathbf{K}$. *Let* p *and* q *be polynomials over* $\mathbf{F}$. *Prove that their greatest common divisor in the ring of polynomials over* $\mathbf{F}$ *is the same as their* gcd *in the ring of polynomials over* $\mathbf{K}$.

Problem 3 *Let* X *be the space of orthogonal real* $n \times n$ *matrices. Let* $v_0 \in \mathbb{R}^n$. *Locate and describe the elements of* X, *where the map*

$$f : X \to \mathbb{R}, \qquad f(A) = \langle v_0, Av_0 \rangle$$

takes its maximum and minimum values.

Problem 4 *Prove that the group of automorphisms of a cyclic group of prime order* p *is cyclic and find its order.*

Problem 5

1. *Give an example of a differentiable function* $f : \mathbb{R} \to \mathbb{R}$ *whose derivative* f' *is not continuous.*
2. *Let* f *be as in Part 1. If* $f'(0) < 2 < f'(1)$, *prove that* $f'(x) = 2$ *for some* $x \in [0, 1]$.

Problem 6 *Let* $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ *be a complex polynomial of degree* $n > 0$. *Prove*

$$\frac{1}{2\pi i} \int_{|z|=R} z^{n-1} |f(z)|^2 \, dz = a_0 \overline{a}_n R^{2n}.$$

Problem 7 *Let f be a continuous complex valued function on $[0,1]$, and define the function g by*

$$g(z) = \int_0^1 f(t)e^{tz}\,dt \qquad (z \in \mathbb{C}\,).$$

Prove that g is analytic in the entire complex plane.

Problem 8 *Let $U \subset \mathbb{R}^n$ be an open set. Suppose that the maph $: U \to \mathbb{R}^n$ is a homeomorphism from U onto $\mathbb{R}^n$, which is uniformly continuous. Prove $U = \mathbb{R}^n$.*

Problem 9 *Prove that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has*

1. *a one-dimensional invariant subspace, and*

2. *a two-dimensional invariant subspace.*

Problem 10 *Find all pairs of C^∞ functions $x(t)$ and $y(t)$ on $\mathbb{R}$ satisfying*

$$x'(t) = 2x(t) - y(t), \qquad y'(t) = x(t).$$

Problem 11 *Let A and B be $n \times n$ matrices over a field $\mathbf{F}$ such that $A^2 = A$ and $B^2 = B$. Suppose that A and B have the same rank. Prove that A and B are similar.*

Problem 12 *Which rational numbers t are such that*

$$3t^3 + 10t^2 - 3t$$

is an integer?

Problem 13 *Let $\mathbf{F}$ be a finite field with q elements and let V be an n-dimensional vector space over $\mathbf{F}$.*

1. *Determine the number of elements in V.*

2. *Let $GL_n(\mathbf{F})$ denote the group of all $n \times n$ nonsingular matrices over $\mathbf{F}$. Determine the order of $GL_n(\mathbf{F})$.*

3. *Let $SL_n(\mathbf{F})$ denote the subgroup of $GL_n(\mathbf{F})$ consisting of matrices with determinant 1. Find the order of $SL_n(\mathbf{F})$.*

Problem 14 *Let A, B, and C be finite abelian groups such that $A \times B$ and $A \times C$ are isomorphic. Prove that B and C are isomorphic.*

Problem 15 *Show that*

$$\sum_{n=0}^{\infty} \frac{z}{(1+z^2)^n}$$

converges for all complex numbers z exterior to the lemniscate

$$\left|1+z^2\right| = 1.$$

Problem 16 *Let $g_n(z)$ be an entire function having only real zeros, $n = 1, 2, \ldots$. Suppose*

$$\lim_{n\to\infty} g_n(z) = g(z)$$

uniformly on compact sets in $\mathbb{C}$, with g not identically zero. Prove that $g(z)$ has only real zeros.

Problem 17 *Let $f : \mathbb{R}^3 \to \mathbb{R}$ be such that*

$$f^{-1}(0) = \{v \in \mathbb{R}^3 \mid \|v\| = 1\}.$$

Suppose f has continuous partial derivatives of orders $\leqslant 2$. Is there a $p \in \mathbb{R}^3$ with $\|p\| \leqslant 1$ such that

$$\frac{\partial^2 f}{\partial x^2}(p) + \frac{\partial^2 f}{\partial y^2}(p) + \frac{\partial^2 f}{\partial z^2}(p) \geqslant 0 \ \ ?$$

Problem 18 *Let E be a three-dimensional vector space over $\mathbb{Q}$. Suppose $T : E \to E$ is a linear transformation and $Tx = y$, $Ty = z$, $Tz = x + y$, for certain $x, y, z \in E$, $x \neq 0$. Prove that x, y, and z are linearly independent.*

Problem 19 *Let $\{f_n\}$ be a sequence of continuous real functions defined $[0, 1]$ such that*

$$\int_0^1 (f_n(y))^2 \, dy \leqslant 5$$

for all n. Define $g_n : [0, 1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 \sqrt{x+y}\, f_n(y)\, dy.$$

1. *Find a constant* $K \geqslant 0$ *such that* $|g_n(x)| \leqslant K$ *for all* n.
2. *Prove that a subsequence of the sequence* $\{g_n\}$ *converges uniformly.*

Problem 20 *Let* $x : \mathbb{R} \to \mathbb{R}$ *be a solution to the differential equation*

$$5x'' + 10x' + 6x = 0.$$

Prove that the function $f : \mathbb{R} \to \mathbb{R}$,

$$f(t) = \frac{x(t)^2}{1 + x(t)^4}$$

attains a maximum value.