Preliminary Exam - Summer 1981

Problem 1 *Let*

$$y(h) = 1 - 2\sin^2(2\pi h), \qquad f(y) = \frac{2}{1+\sqrt{1-y}}.$$

Justify the statement

$$f\left(y(h)\right) = 2 - 4\sqrt{2}\pi\,|h| + O(h^2)$$

where

$$\limsup_{h\to 0} \frac{O(h^2)}{h^2} < \infty.$$

Problem 2 *Let G be a finite group, and let φ be an automorphism of G which leaves fixed only the identity element of G.*

1. *Show that every element of G may be written in the form $g^{-1}\varphi(g)$.*
2. *If φ has order 2 (i.e., $\varphi \cdot \varphi = \mathrm{id}$) show that φ is given by the formula $g \mapsto g^{-1}$ and that G is an abelian group whose order is odd.*

Problem 3 *Prove or disprove: The set $\mathbb{Q}$ of rational numbers is the intersection of a countable family of open subsets of $\mathbb{R}$.*

Problem 4 *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, with*

$$\int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

Show that there is a sequence (x_n) such that $x_n \to \infty$, $x_n f(x_n) \to 0$, and $x_n f(-x_n) \to 0$ as $n \to \infty$.

Problem 5 *Let S denote the vector space of real $n \times n$ skew-symmetric matrices. For a nonsingular matrix A, compute the determinant of the linear map $T_A : S \to S$, $T_A(X) = AXA^t$.*

Problem 6 *Let $S\mathbb{O}(3)$ denote the group of orthogonal transformations of $\mathbb{R}^3$ of determinant 1. Let $Q \subset S\mathbb{O}(3)$ be the subset of symmetric transformations $\neq I$. Let P^2 denote the space of lines through the origin in $\mathbb{R}^3$.*

1. *Show that P^2 and $S\mathbb{O}(3)$ are compact metric spaces (in their usual topologies).*

2. *Show that P^2 and Q are homeomorphic.*

Problem 7 *Compute*

$$\frac{1}{2\pi i}\int_C \frac{dz}{\sin\frac{1}{z}},$$

where C is the circle $|z| = \dfrac{1}{5}$, positively oriented.

Problem 8 *Show that $x^{10}+x^9+x^8+\cdots+x+1$ is irreducible over $\mathbb{Q}$. How about $x^{11}+x^{10}+\cdots+x+1$?*

Problem 9 *Let $f : \mathbb{R} \to \mathbb{R}$ be the function of period 2π such that $f(x) = x^3$ for $-\pi \leqslant x < \pi$.*

1. *Prove that the Fourier series for f has the form $\sum_1^\infty b_n \sin nx$ and write an integral formula for b_n (do not evaluate it).*

2. *Prove that the Fourier series converges for all x.*

3. *Prove*
$$\sum_{n=1}^{\infty} b_n^2 = \frac{2\pi^6}{7}.$$

Problem 10 *Let S be a vector space of complex sequences $\{a_n\}_{n=1}^\infty$. Define the map $T : S \to S$ by $T\{a_1, a_2, a_3, \ldots\} = \{a_2, a_3, \ldots\}$.*

1. *Describe the eigenvectors of T.*

2. *Consider the difference equation $x_{n+2} = x_{n+1} + x_n$. Show that the solutions of this equation define a two-dimensional subspace $E \subset S$ and that $T(E) \subset E$. Find an explicit basis for E.*

3. *The Fibonacci numbers are defined recursively by $f_1 = 1$, $f_2 = 1$, and $f_{n+2} = f_{n+1} + f_n$ for $n \geqslant 1$. Find and explicit formula for f_n.*

Note: See also Problem **??**.

Problem 11 *Show that the equation*

$$x\left(1+\log\left(\frac{1}{\varepsilon\sqrt{x}}\right)\right)=1, \quad x>0, \quad \varepsilon>0\ ,$$

has, for each sufficiently small $\varepsilon > 0$, exactly two solutions. Let $x(\varepsilon)$ be the smaller one. Show that

1. $x(\varepsilon) \to 0$ *as* $\varepsilon \to 0+$;

   yet for any $s > 0$,

2. $\varepsilon^{-s}x(\varepsilon) \to \infty$ *as* $\varepsilon \to 0+$.

Problem 12 *Show that no commutative ring with identity has additive group isomorphic to $\mathbb{Q}/\mathbb{Z}$.*

Problem 13 *Let G be an additive group, and $u, v : G \to G$ homomorphisms. Show that the map $f : G \to G$, $f(x) = x - v(u(x))$ is surjective if the map $h : G \to G$, $h(x) = x - u(v(x))$ is surjective.*

Problem 14 *Let $I \subset \mathbb{R}$ be the open interval from 0 to 1.Let $f : I \to \mathbb{C}$ be C^1 (i.e., the real and imaginary parts are continuously differentiable). Suppose that $f(t) \to 0$, $f'(t) \to C \neq 0$ as $t \to 0+$. Show that the function $g(t) = |f(t)|$ is C^1 for sufficiently small $t > 0$ and that $\lim_{t\to 0+} g'(t)$ exists, and evaluate the limit.*

Problem 15 *Let V be a finite-dimensional vector space over the rationals $\mathbb{Q}$ and let M be an automorphism of V such that M fixes no nonzero vector in V. Suppose that M^p is the identity map on V, where p is a prime number. Show that the dimension of V is divisible by $p - 1$.*

Problem 16 *Let $\{f_n\}$ be a sequence of continuous functions defined from $[0,1] \to \mathbb{R}$ such that*

$$\int_0^1 (f_n(x) - f_m(x))^2\, dx \to 0 \ \ \textit{as} \ \ n, m \to \infty.$$

Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be continuous. Define $g_n : [0,1] \to \mathbb{R}$ by

$$g_n(x) = \int_0^1 K(x,y) f_n(y)\, dy.$$

Prove that the sequence $\{g_n\}$ converges uniformly.

Problem 17 *Suppose that $f(z)$ and $g(z)$ are entire functions such that $|f(z)| \leqslant |g(z)|$ for all z. Show that $f(z) = cg(z)$ for some constant $c \in \mathbb{C}$.*

Problem 18 *Let A and B be square matrices of rational numbers such that $CAC^{-1} = B$ for some real matrix C. Prove that such a C can be chosen to have rational entries.*

Problem 19 *Prove that the number of roots of the equation$z^{2n} + \alpha^2 z^{2n-1} + \beta^2 = 0$ (n a natural number, α and β real, nonzero) that have positive real part is*

1. *n if n is even, and*
2. *$n-1$ if n is odd.*

Problem 20 *Let $y = y(x)$ be a solution of the differential equation $y'' = -|y|$ with $-\infty < x < \infty$, $y(0) = 1$, and $y'(0) = 0$.*

1. *Show that y is an even function.*
2. *Show that y has exactly one zero on the positive real axis.*