Preliminary Exam - Summer 1985

Problem 1 1. *Show that a real* 2×2 *matrix* A *satisfies* $A^2 = -I$ *if and only if*

$$A = \begin{pmatrix} \pm\sqrt{pq-1} & -p \\ q & \mp\sqrt{pq-1} \end{pmatrix}$$

where p *and* q *are real numbers such that* $pq \geqslant 1$ *and both upper or both lower signs should be chosen in the double signs.*

2. *Show that there is no real* 2×2 *matrix* A *such that*

$$A^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1-\varepsilon \end{pmatrix}$$

with $\varepsilon > 0$.

Problem 2 1. *For* $0 \leqslant \theta \leqslant \frac{\pi}{2}$, *show that*

$$\sin\theta \geqslant \frac{2}{\pi}\theta\,.$$

2. *By using Part 1, or by any other method, show that if* $\lambda < 1$, *then*

$$\lim_{R\to\infty} R^\lambda \int_0^{\frac{\pi}{2}} e^{-R\sin\theta}\, d\theta = 0.$$

Problem 3 *Let* A *be a nonsingular real* $n \times n$ *matrix. Prove that there exists a unique orthogonal matrix* Q *and a unique positive definite symmetric matrix* B *such that* $A = QB$.

Problem 4 *Let* G *be a group of order* 120, *let* H *be a subgroup of order* 24, *and assume that there is at least one left coset of* H *(other than* H *itself) which is equal to some right coset of* H. *Prove that* H *is a normal subgroup of* G.

Problem 5 *By the Fundamental Theorem of Algebra, the polynomial* $x^3 + 2x^2 + 7x + 1$ *has three complex roots,* α_1, α_2, *and* α_3. *Compute* $\alpha_1^3 + \alpha_2^3 + \alpha_3^3$.

Problem 6 *Evaluate the integral*

$$\int_0^\infty \frac{x^{a-1}}{x+1}\,dx$$

where a is a complex number. What restrictions must be put on a?

Problem 7 *Let*

$$f(x) = e^{x^2/2}\int_x^\infty e^{-t^2/2}\,dt$$

for $x > 0$.

1. *Show that $0 < f(x) < \frac{1}{x}$.*

2. *Show that $f(x)$ is strictly decreasing for $x > 0$.*

Problem 8 *Let f be a real valued continuous function on a compact interval $[a, b]$. Given $\varepsilon > 0$, show that there is a polynomial p such that$p(a) = f(a)$, $p'(a) = 0$, and $|p(x) - f(x)| < \varepsilon$ for $x \in [a, b]$.*

Problem 9 *Let $u(x)$, $0 \leqslant x \leqslant 1$, be a real valued C^2 function which satisfies the differential equation*

$$u''(x) = e^x u(x).$$

1. *Show that if $0 < x_0 < 1$, then u cannot have a positive local maximum at x_0. Similarly, show that u cannot have a negative local minimum at x_0.*

2. *Now suppose that $u(0) = u(1) = 0$. Prove that $u(x) \equiv 0$, $0 \leqslant x \leqslant 1$.*

Problem 10 *Prove that for each $\lambda > 1$ the equation $z = \lambda - e^{-z}$ in the half-plane $\Re z \geqslant 0$ has exactly one root, and that this root is real.*

Problem 11

1. *Let G be a cyclic group, and let $a, b \in G$ be elements which are not squares. Prove that ab is a square.*

2. *Give an example to show that this result is false if the group is not cyclic.*

Problem 12 *Let A be an $n \times n$ real matrix and A^t its transpose. Show that A^tA and A^t have the same range.*

Problem 13 *Let $P(z)$ be a polynomial of degree $< k$ with complex coefficients. Let $\omega_1, \dots, \omega_k$ be the k^{th} roots of unity in $\mathbb{C}$. Prove that*

$$\frac{1}{k}\sum_{i=1}^{k} P(\omega_i) = P(0).$$

Problem 14 *Let $M_n(\mathbf{F})$ denote the ring of $n \times n$ matrices over a field $\mathbf{F}$. For $n \geqslant 1$, does there exist a ring homomorphism from $M_{n+1}(\mathbf{F})$ onto $M_n(\mathbf{F})$?*

Problem 15 *For each $k > 0$, let X_k be the set of analytic functions $f(z)$ on the open unit disc $\mathbb{D}$ such that*

$$\sup_{z\in\mathbb{D}} \left\{(1-|z|)^k\,|f(z)|\right\}$$

is finite. Show that $f \in X_k$ if and only if $f' \in X_{k+1}$.

Problem 16 *A function $f : [0,1] \to \mathbb{R}$ is said to be upper semicontinuous if given $x \in [0,1]$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that if $|y-x| < \delta$, then $f(y) < f(x) + \varepsilon$. Prove that an upper semicontinuous function f on $[0,1]$ is bounded above and attains its maximum value at some point $p \in [0,1]$.*

Problem 17 *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where all the a_n are nonnegative reals, and the series has radius of convergence 1. Prove that $f(z)$ cannot be analytically continued to a function analytic in a neighborhood of $z = 1$.

Problem 18 *Solve the differential equations*

$$\frac{dy_1}{dx} = -3y_1 + 10y_2,$$
$$\frac{dy_2}{dx} = -3y_1 + 8y_2.$$

Problem 19 *Let $A_1 \geqslant A_2 \geqslant \cdots \geqslant A_k \geqslant 0$. Evaluate*

$$\lim_{n\to\infty} \left(A_1^n + A_2^n + \cdots + A_k^n\right)^{1/n}.$$

Note: See also Problem **??**.

Problem 20 *Let $\mathbf{F}$ be a field of characteristic $p > 0$, $p \neq 3$. If α is a zero of the polynomial $f(x) = x^p - x + 3$ in an extension field of $\mathbf{F}$, show that $f(x)$ has p distinct zeros in the field $\mathbf{F}(\alpha)$.*