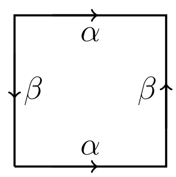
UO TOPOLOGY QUALIFYING EXAM
FALL 2019
SOLUTIONS

(1) Recall that the Klein bottle K can be described as the following identification space:



Show that the Klein bottle retracts onto one of the circles α, β but not the onto the other.

Solution 1. Identify the square above with $[0,1]^2$ in the obvious way. The map $f\colon [0,1]^2 \to [0,1]/(0 \sim 1) = \alpha$ given by $f(x,y) = x$ respects the equivalence relation, hence descends to a continuous map $\bar{f}\colon K \to \alpha$. By definition, $\bar{f}|_\alpha$ is the identity map, so $\bar{f}$ is a retraction.

Next, we show that there is no retraction $r\colon K \to \beta$. If we let H_1 denote the abelianization of π_1, van Kampen's theorem gives $H_1(K) \cong \mathbb{Z}\langle \alpha, \beta\rangle/(2\beta)$ (where we are abusing notation to let α and β denote the elements of $\pi_1(K)$ which go around α and β once). Let $i\colon \beta \hookrightarrow K$ denote inclusion. If there were a retraction $r\colon K \to \beta$, so $r \circ i = \mathbb{I}_\beta$, then we would have

$$r_* \circ i_* = \mathbb{I}\colon H_1(\beta) \to H_1(\beta).$$

But $H_1(\beta) = \mathbb{Z}\langle \beta \rangle$, so $i_*\colon H_1(\beta) \to H_1(K)$ is not injective, a contradiction.

Solution 2 (sketch). The same as above, except use H_1 computed by cellular homology or the Mayer-Vietoris theorem.

(2) Let $M(3,1)$ be the result of attaching a 2-cell to S^1 by the map $z \mapsto z^3$. Describe explicitly, with proof, all connected covering spaces of $M(3,1) \times \mathbb{R}P^2$.

Solution. First, recall that the covering spaces of $X \times Y$ are exactly the products of covering spaces of X and covering spaces of Y. (One can prove this directly, or from the classification of covering spaces and the fact that $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. A proof is not required for full credit for this problem.) Now, the connected covering spaces of $M(3,1)$ are in bijection with subgroups of $\pi_1(M(3,1)) = \mathbb{Z}/3\mathbb{Z}$, of which there are two: $\mathbb{Z}/3\mathbb{Z}$ and $\{0\}$. Similarly, covering spaces of $\mathbb{R}P^2$ are in bijection with subgroups of $\mathbb{Z}/2\mathbb{Z}$, of which the only two are $\mathbb{Z}/2\mathbb{Z}$ and $\{0\}$. Hence, there are four connected covering spaces of $M(3,1) \times \mathbb{R}P^2$.

The two connected covering spaces of $\mathbb{R}P^2$ are $\mathbb{I}\colon \mathbb{R}P^2 \to \mathbb{R}P^2$ and the quotient map $S^2 \to S^2/\{\pm 1\} = \mathbb{R}P^2$.

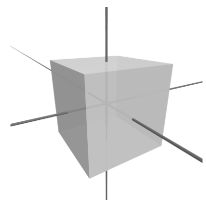
The two connected covering spaces of $M(3,1)$ are $\mathbb{I}\colon M(3,1) \to M(3,1)$ and another one, $f\colon X \to M(3,1)$ defined as follows. Let

$$X = D^2 \times \{0,1,2\}/\sim$$

where $(x,i) \sim (x,j)$ for each $x \in \partial D^2$. Let $q\colon D^2 \to M(3,1)$ be the quotient map. Then $f(x,j) = q(e^{2\pi j\sqrt{-1}/3}x)$. (A clear picture would also suffice here, though $M(3,1)$ does not embed in $\mathbb{R}^3$.)

Now, the four connected covering spaces of $M(3,1)\times\mathbb{R}P^2$ are $M(3,1)\times\mathbb{R}P^2$, $M(3,1)\times S^2$, $X\times\mathbb{R}P^2$, and $X\times S^2$, with the obvious maps.

(3) Let X be the union of the (hollow) cube $\partial([-1,1]^3)$ and the three coordinate axes in $\mathbb{R}^3$



(a) Compute $\pi_1(X)$.
(b) Compute the homology groups of X.

Solution. We start by replacing X by a homotopy equivalent space where the computations are easier. First, X deformation retracts to the union of the hollow cube and the parts of the coordinate axes lying inside the cube. Call the image of this deformation retraction Y. The space Y can be given the structure of a CW complex with, say:

- 0-skeleton $\{(0,0,0),(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)\}$,
- 1-skeleton $Y\cap\{(x,y,z)\mid xyz=0\}$
- 8 2-cells, around the eight vertices of the cube.

(A good picture would be a fine substitute for words here.) Let $Z\subset Y$ be the union of:

- 5 of the 6 faces of the cube, and
- the segment from one of those five faces to $(0,0,0)$.

Then Z is a contractible subcomplex of Y, and the space Y/Z is homeomorphic to the wedge sum of S^2 and 5 circles,

$$S^2\vee S^1\vee S^1\vee S^1\vee S^1\vee S^1.$$

Since both π_1 and H_* are homotopy invariants,

$$\begin{aligned}\pi_1(X)&\cong\pi_1(S^2\vee S^1\vee S^1\vee S^1\vee S^1\vee S^1)\\ H_i(X)&\cong H_i(S^2\vee S^1\vee S^1\vee S^1\vee S^1\vee S^1).\end{aligned}$$

So, it is immediate from van Kampen's theorem and cellular homology that

$$\begin{aligned}&\pi_1(X)\cong *_{i=1}^5\pi_1(S^1)\cong F_5\\ &H_0(X)\cong\mathbb{Z}\\ &H_1(X)\cong\bigoplus_{i=0}^5 H_1(S^1)\cong\mathbb{Z}^5\\ &H_2(X)\cong H_2(S^2)\cong\mathbb{Z}\\ &H_i(X)=0\qquad i>2\end{aligned}$$

(Students do not need to spell out further details here for full credit.)

Solution 2 (sketch). Quotient by a different contractible subcomplex, or apply van Kampen's theorem, and the Mayer-Vietoris sequence or cellular homology, directly to X.

(4) Let $\phi\colon S^2 \times S^2 \to S^2 \times S^2$ be the map $\phi(x,y) = (y,x)$. Let $T_\phi = (S^2 \times S^2 \times [0,1])/((x,y,1) \sim (y,x,0))$ be the mapping torus of ϕ.

(a) Compute the homology groups of T_ϕ.

Solution 1. Let

$$\begin{aligned} U &= T_\phi \setminus (S^2 \times S^2 \times \{0\}) \\ V &= T_\phi \setminus (S^2 \times S^2 \times \{1/2\}). \end{aligned}$$

There is an obvious homeomorphism $f\colon U \xrightarrow{\cong} S^2 \times S^2 \times (0,1)$. There is also a homeomorphism $g\colon V \xrightarrow{\cong} S^2 \times S^2 \times (1/2, 3/2)$ defined by

$$g(p,t) = \begin{cases} (\phi^{-1}(p), t+1) & 0 \le t < 1/2 \\ (p,t) & 1/2 < t \le 1. \end{cases}$$

For $I = (0,1)$, $I = (1/2,3/2)$, $I = (0,1/2)$, or $I = (1/2,1)$, let $p\colon S^2 \times S^2 \times I \to S^2 \times S^2$ denote projection. Then we have isomorphisms

$$\begin{aligned} (p \circ f)_*\colon H_*(U) &\xrightarrow{\cong} H_*(S^2 \times S^2) \\ (p \circ g)_*\colon H_*(V) &\xrightarrow{\cong} H_*(S^2 \times S^2) \\ ((p \amalg p) \circ f)_*\colon H_*(U \cap V) &\xrightarrow{\cong} H_*(S^2 \times S^2) \oplus H_*(S^2 \times S^2). \end{aligned}$$

Apply the Mayer-Vietoris theorem to the cover $T_\phi = U \cup V$ and use the identifications above to obtain

$$\begin{tikzcd} \cdots \arrow[r] & H_i(U \cap V) \arrow[r] \arrow[d, "((p\amalg p)\circ f)_*"'] & H_i(U) \oplus H_i(V) \arrow[r] \arrow[d, "(p\circ f)_* \oplus (p \circ g)_*"'] & H_i(T_\phi) \arrow[r] & \cdots \\ & H_i(S^2 \times S^2) \oplus H_i(S^2 \times S^2) \arrow[r, "\Psi_i"'] & H_i(S^2 \times S^2) \oplus H_i(S^2 \times S^2). & & \end{tikzcd}$$

The map Ψ is the unique map so that the diagram commutes. It follows from the definitions that

$$\Psi_i = \begin{pmatrix} \mathbb{I} & \mathbb{I} \\ (\phi^{-1})_* & \mathbb{I} \end{pmatrix}.$$

(Depending on one's sign convention for the Mayer-Vietoris sequence, there might be minus signs in the second row.) There is a short exact sequence

$$0 \to \operatorname{coker}(\Phi_i) \to H_i(T_\phi) \to \ker(\Phi_{i-1}) \to 0.$$

Note that $\phi^2 = \mathbb{I}$, so $(\phi^{-1})_* = \phi_*$. Also, row-reducing,

$$\begin{aligned} \ker(\Phi_i) &= \ker(\phi_* - \mathbb{I}) \\ \operatorname{coker}(\Phi_i) &= \operatorname{coker}(\phi_* - \mathbb{I}). \end{aligned}$$

From cellular homology (or the Künneth theorem), we have

$$H_i(S^2 \times S^2) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}^2 & i = 2 \\ \mathbb{Z} & i = 4 \\ 0 & \text{else.} \end{cases}$$

Further, by considering degrees, ϕ_* is the identity map on H_0, the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on H_2, and the identity map on H_4.

Hence, we have

$$\begin{aligned}
H_0(T_\phi) &\cong \mathrm{coker}\big((\phi_* - \mathbb{I})\colon H_0(S^2\times S^2) \to H_0(S^2\times S^2)\big) \cong \mathbb{Z}\\
H_1(T_\phi) &\cong \ker\big((\phi_* - \mathbb{I})\colon H_0(S^2\times S^2) \to H_0(S^2\times S^2)\big) \cong \mathbb{Z}\\
H_2(T_\phi) &\cong \mathrm{coker}\big((\phi_* - \mathbb{I})\colon H_2(S^2\times S^2) \to H_2(S^2\times S^2)\big) \cong \mathrm{coker}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cong \mathbb{Z}\\
H_3(T_\phi) &\cong \ker\big((\phi_* - \mathbb{I})\colon H_2(S^2\times S^2) \to H_2(S^2\times S^2)\big) \cong \ker\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \cong \mathbb{Z}\\
H_4(T_\phi) &\cong \mathrm{coker}\big((\phi_* - \mathbb{I})\colon H_4(S^2\times S^2) \to H_4(S^2\times S^2)\big) \cong \mathbb{Z}\\
H_5(T_\phi) &\cong \ker\big((\phi_* - \mathbb{I})\colon H_4(S^2\times S^2) \to H_4(S^2\times S^2)\big) \cong \mathbb{Z}
\end{aligned}$$

Solution 2. (sketch). Hatcher gives a long exact sequence for the homology of a mapping torus, which we did not cover in class but which some students might know.

Solution 3. (sketch). It is a bit tedious, but this computation can be done using cellular homology.

(5) (a) Define the compactly supported cohomology groups H^i_c of a space X.

Solution. If $K \subset L \subset X$ then $X \setminus K \supset X \setminus L$. Hence, the inclusion map of pairs $(X, X\setminus L) \hookrightarrow (X, X\setminus K)$ induces a map of relative cohomology $H^i(X, X\setminus K) \to H^i(X, X\setminus L)$. Further, if $K \subset L \subset M$ then, since the diagram of inclusions

$$\begin{tikzcd}
(X, X\setminus K) & & (X, X\setminus M) \arrow[ll] \arrow[dl] \\
& (X, X\setminus L) \arrow[ul] &
\end{tikzcd}$$

commutes, the diagram of relative cohomologies

$$\begin{tikzcd}
H^i(X, X\setminus K) \arrow[rr] \arrow[dr] & & H^i(X, X\setminus M) \\
& H^i(X, X\setminus L) \arrow[ur] &
\end{tikzcd}$$

commutes.

Hence, the groups

$$\{H^i(X, X\setminus K)\}_{K\subset X \text{ compact}}$$

form a directed system. The compactly supported cohomology $H^i_c(X)$ is the direct limit of this directed system.

Solution 2 (sketch). Alternatively, one can define

$$C^i_c(X) = \varinjlim_{K\subset X \text{ compact}} C^i(X, X\setminus K),$$

see that d induces a map $d\colon C^i_c(X) \to C^{i+1}_c(X)$ and these maps form a chain complex, and define $H^i_c(X)$ to be the homology of this chain complex.

(b) Show that H_c^i is not a cohomology theory. More precisely, show that there is no cohomology theory h^* so that $h^i(X) \cong H_c^i(X)$ for all spaces X and integers i.
Solution. If h^* is a cohomology theory then the homotopy axiom implies that if $X \simeq Y$ then $h^i(X) \cong h^i(Y)$ for all i. For compactly supported cohomology, by definition $H_c^0(\mathbb{R}^0) = H^0(\mathbb{R}^0, \emptyset) \cong \mathbb{Z}$. On the other hand, $H_c^0(\mathbb{R}^1) = 0$: it follows from Poincaré duality that $H_c^0(\mathbb{R}^1) \cong H_1(\mathbb{R}^1) = 0$. (Alternatively, it is not hard to show directly that $H_c^0(\mathbb{R}^1) = 0$.)
Remark. Compactly-supported cohomology is functorial under proper maps (though not all maps), and invariant under proper homotopies.

(6) Let $(\mathbb{R}P^2)^{2019}$ be the product of 2019 copies of $\mathbb{R}P^2$ with itself. Suppose $f\colon (\mathbb{R}P^2)^{2019} \to (\mathbb{R}P^2)^{2019}$ is a continuous. Show f has a fixed point.
Solution. Recall that the homology of $\mathbb{R}P^2$ is

$$H_i(\mathbb{R}P^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \mathbb{Z}/2\mathbb{Z} & i=1 \\ 0 & \text{otherwise.} \end{cases}$$

(This follows easily, for example, from cellular homology, or from the long exact sequence for a pair or the Mayer-Vietoris sequence.) Hence, by the universal coefficient theorem,

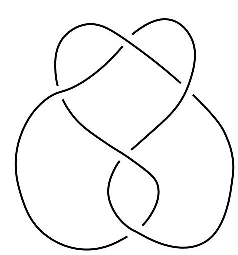
$$H_i(\mathbb{R}P^2;\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i=0 \\ 0 & \text{otherwise.} \end{cases}$$

Now, by the Künneth theorem,

$$H_i((\mathbb{R}P^2)^{2019};\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i=0 \\ 0 & \text{otherwise.} \end{cases}$$

For any map $f\colon X \to X$, $f_*\colon H_0(X) \to H_0(X)$ is the identity map. Hence, for any map $f\colon (\mathbb{R}P^2)^{2019} \to (\mathbb{R}P^2)^{2019}$, the Lefschetz trace $\tau(f) = 1$. Hence, f has a fixed point.

(7) Consider the knot 5_2



(a) I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/537\mathbb{Z}$. Do you believe me? Justify.
(b) Now I claim I have a normal covering space of $S^3 \setminus 5_2$ with deck transformation group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Do you believe me? Justify.

Solution 1. From the classification of covering spaces, a space X has a normal covering space $\widetilde{X}$ with deck group G if and only if $\pi_1(X)$ has a normal subgroup H with $\pi_1(X)/H \cong G$. Further, if G is abelian then H must contain the commutator subgroup of $\pi_1(X)$, so

$$H_1(X) = \pi_1(X)/[\pi_1(X), \pi_1(X)] \twoheadrightarrow \pi_1(X)/H \cong G.$$

Conversely, if $H_1(X)$ surjects onto G then $\ker(\pi_1(X) \to H_1(X) \to G)$ corresponds to a normal covering space with deck group G.

By Alexander duality, $H_1(S^3 \setminus 5_2) \cong H^1(S^1) \cong \mathbb{Z}$. Since $\mathbb{Z}$ surjects onto $\mathbb{Z}/537\mathbb{Z}$, $S^3 \setminus 5_2$ does have a normal covering space with deck group $\mathbb{Z}/537\mathbb{Z}$. Since $\mathbb{Z}$ does not surject onto $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $S^3 \setminus 5_2$ does not have a normal covering space with deck group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Solution 2 (sketch). Students in the class have seen the Wirtinger presentation for $\pi_1(S^3 \setminus K)$ and could substitute that for Alexander duality (though it is slightly tedious).

(8) Let G be a finitely generated abelian group. Show that no closed 3-manifold is a $K(G,2)$. (Hint: reduce to the orientable case and consider the homology of a $K(G,2)$.)

Solution 1. Let M be a closed 3-manifold. If M is non-orientable then the orientation double cover of M is a nontrivial 2-fold cover, so $\pi_1(M) \neq 0$, so M is not a $K(G,2)$.

Next, if M is orientable then by Poincaré duality, $H_3(M) \cong \mathbb{Z}$. So, it suffices to show that $H_3(K(G,2)) = 0$. We can build a space $K(G,2)$ as follows. Start with a Moore space built from 2-cells and 3-cells. By cellular approximation (or van Kampen's theorem), $\pi_1(M(G,2)) = 0$, and by the Hurewicz theorem, $\pi_2(M(G,2)) \cong H_2(M(G,2)) \cong G$. Now, attach 4-cells to $M(G,2)$ to kill of $\pi_3(M(G,2))$, attach 5-cells to the result to kill of π_4, and so on. Since the resulting $K(G,2)$ has the same 3-skeleton as $M(G,2)$, $H_3(K(G,2))$ is a quotient of $H_3(M(G,2)) = 0$, hence vanishes. In particular, $H_3(M) \not\cong H_3(K(G,2))$.

Solution 2. Suppose that M is a $K(G,2)$. As in Solution 1, M is orientable. By the 1-dimensional Hurewicz theorem, $H_1(M) = 0$, so by Poincaré duality $H^2(M) = 0$, so by the universal coefficient theorem $H_2(M) = 0$. So, by the Hurewicz theorem one more time, $G = \pi_2(M) = 0$. Now, if M is a $K(\{0\},2)$ then $\pi_i(M) = 0$ for all i so by the Hurewicz theorem $H_i(M) = 0$ for all $i > 0$. In particular, $H_3(M) = 0$, which contradicts the fact that M was closed and orientable.

(9) Recall that orientable k-dimensional vector bundles over X are in bijection with $[X, \mathrm{Gr}_3^+(\mathbb{R}^\infty)]$, where $\mathrm{Gr}_3^+(\mathbb{R}^\infty) = V_3(\mathbb{R}^\infty)/SO(3)$ is the Grassmanian of oriented 3-planes in $\mathbb{R}^\infty$. Compute $\pi_i(\mathrm{Gr}_3^+(\mathbb{R}^\infty))$ for $i \leq 4$. (Hint: recall that $SO(3) \cong \mathbb{R}P^3$.)

Solution. The space $V_3(\mathbb{R}^\infty)$ is contractible, so the long exact sequence for the fibration $SO(3) \to V_3(\mathbb{R}^\infty) \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ decomposes as

$$0 = \pi_n(V_3(\mathbb{R}^\infty) \to \pi_n(\mathrm{Gr}_3^+(\mathbb{R}^\infty)) \to \pi_{n-1}(SO(3)) \to \pi_{n-1}(V_3(\mathbb{R}^\infty)) = 0.$$

Hence, $\pi_n(\mathrm{Gr}_3^+(\mathbb{R}^\infty)) \cong \pi_{n-1}(SO(3))$.

As noted in the hint, $SO(3) \cong \mathbb{R}P^3$. Hence, $\pi_1(SO(3)) \cong \mathbb{Z}/2\mathbb{Z}$ and for $i > 1$, $\pi_i(SO(3)) \cong \pi_i(S^3)$ (since S^3 is a covering space of $\mathbb{R}P^3$). From the Hurewicz theorem, $\pi_2(S^3) = 0$ and $\pi_3(S^3) \cong H_3(S^3) \cong \mathbb{Z}$. Hence, the first few homotopy groups of $\mathrm{Gr}_3^+(\mathbb{R}^\infty))$ are:

$$\pi_i(\mathrm{Gr}_3^+(\mathbb{R}^\infty)) = \begin{cases} 0 & i = 0 \\ 0 & i = 1 \\ \mathbb{Z}/2\mathbb{Z} & i = 2 \\ 0 & i = 3 \\ \mathbb{Z} & i = 4. \end{cases}$$

(10) Let Y be a 2-connected space and $p\colon Y \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ a fibration so that $p_*\colon \pi_i(Y) \to \pi_i(\mathrm{Gr}_3^+(\mathbb{R}^\infty))$ is an isomorphism for $i > 2$. (That is, Y is a 2-*connected cover* of $\mathrm{Gr}_3^+(\mathbb{R}^\infty)$.) Define the (primary) obstruction in cohomology to lifting a map $f\colon X \to$

$\mathrm{Gr}_3^+(\mathbb{R}^\infty)$ to a map $\widetilde{f}\colon X \to Y$ and give an example where the obstruction does not vanish.

Solution. From the long exact sequence in homotopy groups, the fibration $Y \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ has fiber $K(\mathbb{Z}/2\mathbb{Z}, 1)$. Since $\pi_1(Y) \cong \pi_1(\mathrm{Gr}_3^+(\mathbb{R}^\infty)) = 0$, the map $Y \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ is a principal fibration. (This is a special case of the statement about Moore-Postnikov fibrations on the "possibly useful theorems" page, and is also immediate from the construction above.) So, a map $f\colon X \to \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ has a lift if and only if the composite

$$X \xrightarrow{f} \mathrm{Gr}_3^+(\mathbb{R}^\infty) \xrightarrow{g} K(\mathbb{Z}/2\mathbb{Z}, 2)$$

is nullhomotopic. The homotopy class $[g \circ f] \in [X, K(\mathbb{Z}/2\mathbb{Z}, 2)]$ is trivial if and only if

$$(g \circ f)^*(\iota) \in H^2(X; \mathbb{Z}/2\mathbb{Z})$$

vanishes. The element $(g \circ f)^*(\iota)$ is the primary obstruction to lifting f.

For an example where the primary obstruction does not vanish, take $X = \mathrm{Gr}_3^+(\mathbb{R}^\infty)$ and let f be the identity map. Then from the construction in the previous solution $(g \circ \mathbb{I})^*(\iota) = g^*(\iota)$ is a generator of $H^2(\mathrm{Gr}_3^+(\mathbb{R}^\infty); \mathbb{Z}/2\mathbb{Z})$.