Topology Qualifying Exam Workshop

May 2020

Worksheet 1

Part 1 - 5-10 minutes

1. Discuss with your group: What topics on the qual do you feel confident about? What topics are you less confident about? *(The syllabus can be found at `https://www.math.unl.edu/graduate/exams/quals/topology/871-872Qualifier_Syllabus.pdf`.)*
2. Consider for yourself, and share with your group if you feel comfortable: Are you excited for the exam? Anxious? Worried?

Part 2 - 1 hour and 40 minutes

This was the qual that I took. This is not a reasonable amount of time to finish these problems, but look them all over, and pick and choose a few to try.

Do three problems from:

1. (May 2016) Given any topological space Z and subset $D \subseteq Z$, let $Cl_Z(D)$ denote the closure of D in Z. Show that if X and Y are topological spaces and $A \subseteq X$, $B \subseteq Y$, then $Cl_{X\times Y}(A \times B) = Cl_X(A) \times Cl_Y(B)$.
2. (May 2016) Let X be a connected space and $A, B \subseteq X$ be closed subsets of X with $X = A \cup B$ and $A \cap B$ a connected subset of X. Show that both A and B are connected.
3. (May 2016) Let X be the set of real numbers, let $\mathcal{T}_E$ be the Euclidean topology on X, and let $\mathcal{T}_0$ be the excluded point topology (that is, $\mathcal{T}_0 = \{U \subset X | 0 \notin U\} \cup \{X\}$). For each of the following topological spaces, determine whether or not the space is compact:
   (a) The set X with the topology $\mathcal{T}_E \cap \mathcal{T}_0$.
   (b) The set X with the topology generated by the subbasis $\mathcal{T}_E \cup \mathcal{T}_0$.
4. (May 2016) Suppose that the space X has the fixed point property (that is, for any continuous function $f : X \to X$ there is a point $p \in X$ with $f(p) = p$). Suppose also that $A \subset X$ is a subspace admitting a retraction $r : X \to A$. Show that A also has the fixed point property.

Do three problems from:

1. (May 2016) Let $X = S^1 \times S^1$, also thought of as the standard quotient of the unit square $[0, 1] \times [0, 1]$, and let $A = (x, x) : x \in S^1\}$ be the diagonal of X. Show that A is a retract of X, but not a deformation retract of X.
2. (May 2016) A group G is called *residually finite* if for every $g \in G$ with $g \neq 1$, there is a finite group H and a (surjective) homomorphism $\varphi; G \to H$ with $\varphi(g) \neq 1$. Let G be a residually finite group and let X be the presentation complex for a presentation of G, with vertex x_0. Show that for any loop $\gamma : I \to X$ at x_0 with $1 \neq [\gamma] \in \pi_1(X, x_0)$, there is a finite-sheeted covering space $p : \tilde{X} \to X$ and a basepoint $\tilde{x}_0 \in p^{-1}(\{x_0\})$ such that γ does not lift to a loop at $\tilde{x}_0$.
3. (May 2016) Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be covering spaces of path-connected, locally path-connected spaces X and Y with $\tilde{X}$ and $\tilde{Y}$ locally path-connected and simply connected. Show that if X and Y are homeomorphic, then $\tilde{X}$ and $\tilde{Y}$ are homeomorphic.
4. (May 2016) Construct a Δ-complex structure, and use it to compute the simplicial homology groups, for the connected sum of two projective planes.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Are these problems roughly what you expected? Harder? Easier? More technical?
2. What have you learned from these problems? *Your answer doesn't have to be strictly mathematical.*