Topology Qualifying Exam Workshop

May 2020

Worksheet 2

Theme: Topological spaces and continuous functions: Topology, open and closed sets, basis, sub- basis; continuous function, homeomorphism; closure, limit points; subspace topology, product topology, and quotient/identification topology.

Also some compactness because there weren't a ton of problems on just those topics.

Part 1 - 5-10 minutes

Warm-up:

1. What is a topology on a set?
2. Describe how to make a topology from a basis, and what (if any) requirements there are on the basis.
3. Describe how to make a topology from a subbasis, and what (if any) requirements there are on the subbasis.
4. Give 4-6 different topologies on $\mathbb{R}$.

Part 2 - 1 hour and 40 minutes

Try these problems:

1. (June 2010) Let X be a topological space and let $f, g : X \to \mathbb{R}$ be continuous functions.
   (a) Show that the set $L = \{p \in X : f(p) \leq g(p)\}$ is a closed subset of X.
   (b) Show that the function $h : X \to \mathbb{R}$ given by $h(p) = \min\{f(p), g(p)\}$ is continuous.
2. (June 2011) Prove that if $f : X \to Y$ is a continuous map between topological spaces and C is a compact subset of X, then $f(C)$ is a compact subset of Y.
3. (June 2012) Suppose X and Y are topological spaces and $f : X \to Y$ is a continuous function.
   (a) Prove there exists a topological space Z, a quotient map $q : X \to Z$ and a one-to-one continuous map $h : Z \to Y$ such that $f = h \circ q$.
   (b) (This part is a continuous of part (a).) If Z_1 is a topological space, $q_1 : X \to Z_1$ is a quotient map, $h_1 : Z_1 \to Y$ is one-to-one and continuous, and $f = h_1 \circ q_1$, prove that Z and Z_1 are homeomorphic.
4. (May 2015) Let $f : (X, \mathcal{T}) \to (Y, \mathcal{T}')$ be a continuous map. A *section* of f is a continuous map $\sigma : (Y, \mathcal{T}') \to (X, \mathcal{T})$ such that $f \circ \sigma = \mathrm{id}_Y$. Show that if f has a section, then f is a quotient map.
5. (May 2020) Let $\mathcal{B} = \{[a, b) : a, b \in \mathbb{R}\}$; let $\mathcal{R}_\ell$ denote $\mathbb{R}$ equipped with the topology generated by the basis $\mathcal{B}$ (you do not need to prove $\mathcal{B}$ is a basis). Decide (with proof) if the following sets are compact in $\mathbb{R}_\ell$:
   (a) $A = \{1/n : n \in \mathbb{N}\} \cup \{0\}$.
   (b) $B = \{-1/n : n \in \mathbb{N}\} \cup \{0\}$.

Part 3 - 5-10 minutes

Wrap-up discussion with everyone:

1. Which of these problems did you find easy? Hard?
2. Which topics in the theme do you need to review more?