Week 2: Calculus II (Part 1)
Practice Problem Solutions

Problem 1. What is the length of the curve $(x(t), y(t)) = (\cos(t), \sin(t))$ for $0 \le t \le \pi$?

Solution. The length is half the circumference of a unit circle to it is π. Alternatively, using the arc length formula:

$$L = \int_0^\pi \sqrt{x'(t)^2 + y'(t)^2}dt = \int_0^\pi \sqrt{\sin^2(t) + \cos^2(t)}dt = \pi.$$

Problem 2. Compute $\displaystyle\int_{e^{-3}}^{e^{-2}} \frac{dx}{x\log(x)}$.

Solution. Using the substitution $y = \log(x)$ gives

$$\int_{e^{-3}}^{e^{-2}} \frac{dx}{x\log(x)} = \int_{-3}^{-2} \frac{dy}{y} = \log|-2| - \log|-3| = \log\left(\tfrac{2}{3}\right).$$

Problem 3. For $n \in \mathbb{N}$, evaluate $\displaystyle\int_0^\infty x^n e^{-x} dx$.

Solution. Defining

$$I_n = \int_0^\infty x^n e^{-x} dx,$$

we see $I_0 = 1$ and for $n \ge 1$,

$$I_n = [-x^n e^{-x}]_{x=0}^{x\to\infty} + n\int_0^\infty x^{n-1}e^{-x}dx = nI_{n-1}.$$

Thus by induction, it is easily seen that $I_n = n!$. (One may recognize that $I_n = \Gamma(n+1)$ where Γ is the Gamma Function)

Problem 4. Perform the integral $\displaystyle\int_{-\infty}^{x} \frac{dt}{\cosh(t)}$. (Recall $\cosh(t) = \frac{e^t + e^{-t}}{2}$)

Solution. We see

$$\int_{-\infty}^{x} \frac{dt}{\cosh(t)} = \int_{-\infty}^{x} \frac{2e^t dt}{1 + e^{2t}} = 2\int_0^{e^x} \frac{ds}{1+s^2} = 2\arctan(e^x)$$

where we made the substitution $s = e^t$.

Note: this function (shifted by a constant) is called the Gudermannian function and gives a connection between the ordinary trig. functions and hyperbolic trig. functions that doesn't invoke complex numbers.

Problem 5. Compute $\displaystyle\int \frac{x+2}{x^3-x^2+2x-2}dx$.

Solution. The denominator factors like $x^3-x^2+2x-2=(x-1)(x^2+2)$. Performing partial fractions, we have

$$\frac{x+2}{(x-1)(x^2+2)} = \frac{A}{x-1}+\frac{Bx+C}{x^2+2} \iff x+2 = A(x^2+2)+(Bx+C)(x-1).$$

Solving gives $A=1, B=-1, C=0$. Thus

$$\int \frac{x+2}{x^3-x^2+2x-2}dx = \int\left(\frac{1}{x-1}-\frac{x}{x^2+2}\right)dx = \log(x-1)-\frac{1}{2}\log(x^2+2) + \text{constant}$$

Problem 6. Evaluate $\displaystyle\int_0^a \frac{x^2+b^2}{x^2+a^2}dx$ where $a,b>0$ are constant.

Solution. Notice that

$$\int_0^a \frac{x^2+b^2}{x^2+a^2}dx = \int_0^a\left(1+\frac{b^2-a^2}{x^2+a^2}\right)dx = a+\left(\frac{b^2-a^2}{a}\right)\arctan\left(\frac{x}{a}\right)\Bigg|_{x=0}^{x=a} = a+\frac{\pi}{4}\left(\frac{b^2-a^2}{a}\right).$$

Problem 7. What volume is created if the area between $f(x)=x$ and $g(x)=x^2$ for $x\in[0,1]$ is revolved about the x-axis? What if the same area is revolved about the y-axis?

Solution. The area occurs on the on the interval $[0,1]$. Thus the volume created when it is revolved about the x-axis is

$$\pi\int_0^1(x^2-x^4)dx = \pi\left(\tfrac{1}{3}-\tfrac{1}{5}\right) = \tfrac{2\pi}{15}$$

and the volume when it is revolved about the y-axis is

$$\pi\int_0^1(y-y^2)dy = \pi\left(\tfrac{1}{2}-\tfrac{1}{3}\right) = \tfrac{\pi}{6}.$$

Problem 8. Compute $\displaystyle\int_0^{\pi/2}\frac{dx}{1+\tan(x)^{2020}}$.

Solution. Call the integral I. Making the substitution $x=\pi/2-y$, we see

$$I = \int_0^{\pi/2}\frac{dx}{1+\tan(x)^{2020}} = \int_0^{\pi/2}\frac{dy}{1+\tan(\pi/2-y)^{2020}}.$$

But $\cos(\pi/2-y)=\sin(y)$ and $\sin(\pi/2-y)=\cos(y)$ so

$$I = \int_0^{\pi/2}\frac{dy}{1+\cot(y)^{2020}} = \int_0^{\pi/2}\frac{\tan(y)^{2020}dy}{1+\tan(y)^{2020}}.$$

Taking this representation of I and adding it to the original, we see

$$2I = \int_0^{\pi/2} \left(\frac{1+\tan(x)^{2020}}{1+\tan(x)^{2020}} \right) dx = \frac{\pi}{2} \quad \Longrightarrow \quad I = \frac{\pi}{4}.$$

Note that curiously enough, this manipulation did not depend on the number 2020 in any way; that is, the integral

$$I(\alpha) = \int_0^{\pi/2} \frac{dx}{1+\tan(x)^{\alpha}}$$

is identically equal to $\pi/4$ for $\alpha \geq 0$.

Problem 9. Compute $\displaystyle\int_0^\infty \frac{\log(t)}{1+t^2} dt$.

Solution. Using the substitution $t \mapsto 1/t$ for $t \in (0,1)$, we see

$$\int_0^1 \frac{\log(t)}{1+t^2} dt = \int_\infty^1 \frac{\log(1/t)}{1+\frac{1}{t^2}} \left(-\frac{1}{t^2}\right) dt = -\int_1^\infty \frac{\log(t)}{1+t^2} dt.$$

Thus the integral is zero since the contributions from $(0,1)$ and $(1,\infty)$ cancel.

Problem 10. (Gabriel's Horn) Let $f(x) = 1/x$, for $x \in [1,\infty)$. Find the volume and surface area of the shape which results from rotating the graph of f about the x-axis.

Solution. The volume is given by

$$V = \pi \int_1^\infty \frac{dx}{x^2} = \pi \left(-\frac{1}{x}\right)\bigg|_{x=1}^{x\to\infty} = \pi.$$

The surface area formula gives

$$SA = 2\pi \int_1^\infty \frac{1}{x}\sqrt{1+\frac{1}{x^2}} dx \geq 2\pi \int_1^\infty \frac{dx}{x} = +\infty$$

so this shape has finite volume but infinite surface area.

Problem 11. Evalutate $\lim\limits_{n\to\infty} (3^n + 5^n)^{1/n}$. More generally, if $x_1, \ldots, x_k > 0$, evaluate the limit $\lim\limits_{n\to\infty} (x_1^n + \ldots + x_k^n)^{1/n}$.

Solution. We see

$$5 \leq (3^n + 5^n)^{1/n} \leq (5^n + 5^n)^{1/n} = 2^{1/n}5.$$

Taking the limit as $n \to \infty$, the squeeze theorem shows that

$$\lim_{n\to\infty} (3^n + 5^n)^{1/n} = 5.$$

More generally
$$\lim_{n\to\infty} (x_1^n + \ldots + x_k^n)^{1/n} = \max\{x_1, \ldots, x_k\}$$
using similar reasoning.

Problem 12. For what values of $\alpha, \beta \in \mathbb{R}$ does the series
$$\sum_{n=2}^{\infty} \frac{1}{n^\alpha \log(n)^\beta}$$
converge/diverge?

Solution. If $\alpha > 1$, then we can compare this series with $\sum \frac{1}{n^\alpha}$ to see that it converges.

If $\alpha < 1$, then we can find $\varepsilon > 0$ small enough that $\alpha + \varepsilon < 1$. Since any power of $\log(n)$ is asymptotically smaller than any power of n, we see that $n^\alpha \log(n)^\beta \lesssim n^{\alpha+\varepsilon}$ and so we can compare this series to $\sum \frac{1}{n^{\alpha+\varepsilon}}$ to see that it diverges.

If $\alpha = 1$, we can use the integral test. Note that
$$\int_2^\infty \frac{dx}{x \log(x)^\beta} = \int_{\log(2)}^\infty \frac{dy}{y^\beta}$$
converges if and only if $\beta > 1$. Thus the series also converges if and only if $\beta > 1$.

Problem 13. Do the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\log(n!)}$ and $\displaystyle\sum_{n=3}^{\infty} \frac{1}{\log(n)^{\log(n)}}$ converge or diverge?

Solution. Using $\log(n!) = \sum_{k=1}^n \log(k) \le n \log(n)$, we have
$$\sum_{n=2}^{\infty} \frac{1}{\log(n!)} \ge \sum_{n=2}^{\infty} \frac{1}{n \log(n)}$$
and so this series diverges by comparison using the result of **Problem 12**.

The other series converges. Indeed, we see that
$$\log(n)^{\log(n)} = e^{\log(\log(n))\log(n)} = \left(e^{\log(n)}\right)^{\log(\log(n))} = n^{\log(\log(n))}.$$
Now for $n > e^{e^2}$, we have $\log(\log(n)) > 2$, thus
$$\sum_{n=3}^{\infty} \frac{1}{\log(n)^{\log(n)}} \le C + \sum_{n=\lceil e^{e^2} \rceil}^{\infty} \frac{1}{n^2} < \infty.$$

Problem 14. Do the series $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^{n^2}}$ and $\displaystyle\sum_{n=1}^{\infty} \frac{n^{\sqrt{n}}}{2^n}$ converge or diverge?

Solution. For the first we use the ratio test. Since

$$\lim_{n\to\infty} \frac{(n+1)!}{n!} \frac{2^{n^2}}{2^{n^2+2n+1}} = \lim_{n\to\infty} \frac{n+1}{2^{2n+1}} = 0$$

the first series converges. For the second series, we use the root test. We have

$$\lim_{n\to\infty} \left(\frac{n^{\sqrt{n}}}{2^n}\right)^{1/n} = \frac{1}{2} \lim_{n\to\infty} n^{1/\sqrt{n}} =: \frac{1}{2}L.$$

Notice that

$$\log L = \lim_{n\to\infty} \frac{\log(n)}{\sqrt{n}} = 0 \quad \implies \quad L = 1$$

and thus the series converges since the root test results in a limit of $1/2$.

Problem 15. Fix an integer $m > 0$. Evaluate the infinite sum

$$\sum_{n=1}^{\infty} \frac{m}{n(n+m)}.$$

Solution. Using partial fractions gives

$$\frac{m}{n(n+m)} = \left(\frac{1}{n} - \frac{1}{n+m}\right).$$

Now when we sum, there will be telescoping so that all terms past $\frac{1}{m}$ cancel, leaving behind

$$\sum_{n=1}^{\infty} \frac{m}{n(n+m)} = \sum_{k=1}^{m} \frac{1}{k}.$$

Problem 16. Decide whether the following series converge or diverge:

$$\text{(a)} \sum_{n=1}^{\infty} [1 - \tanh(n)], \qquad \text{(b)} \sum_{n=1}^{\infty} \left(\frac{\pi}{2} - \arctan(n)\right).$$

Solution. We see

$$1 - \tanh(n) = 1 - \frac{e^n - e^{-n}}{e^n + e^{-n}} = \frac{2e^{-n}}{e^n + e^{-n}} \le 2e^{-2n}$$

and so the first series converges by comparison to the goemetric series $\sum (e^{-2})^n$.

For the second series, consider

$$\lim_{n\to\infty} \frac{\pi/2 - \arctan(n)}{1/n} = \lim_{n\to\infty} \frac{-\frac{1}{1+n^2}}{-1/n^2} = 1$$

and so $(\pi/2 - \arctan(n)) \sim \frac{1}{n}$ which shows that the second series diverges.

Problem 17. Find a sequence (a_n) such that $a_n > 0$ for all $n \in \mathbb{N}$ and $a_n \to 0$ but

$$\sum_{n=1}^{\infty} (-1)^n a_n \quad \text{diverges.}$$

[Note: this shows that the assumption that a_n is decreasing is necessary in the Alternating Series Test.] Find a sequence (b_n) such that

$$\sum_{n=1}^{\infty} b_n \quad \text{converges while} \quad \sum_{n=1}^{\infty} b_n^2 \quad \text{diverges.}$$

Is it possible to choose (b_n) so that $\sum_{n=1}^{\infty} b_n$ converges absolutely while $\sum_{n=1}^{\infty} b_n^2$ diverges?

Solution. For the first part, take $a_{2m-1} = \frac{1}{2^m}$ and $a_{2m} = \frac{1}{m}$. Then clearly $a_n > 0$ and $a_n \to 0$ but the even partial sums are given by

$$\sum_{n=1}^{2N} (-1)^n a_n = \sum_{n=1}^{N} \frac{1}{n} - \sum_{n=1}^{N} \frac{1}{2^n} \geq -1 + \frac{1}{2} \sum_{n=1}^{N} \frac{1}{n} \to \infty \quad \text{as} \quad N \to \infty$$

[where we've used $\sum_{n=1}^{\infty} 2^{-n} = 1$]. Thus the infinite sum does not converge.

For the second part, let $b_n = (-1)^n/\sqrt{n}$. Then $\sum b_n$ converges by the alternating series test but $\sum b_n^2$ is the harmonic series which diverges.

To answer the last question: no, this is impossible. If b_n converges, then $b_n \to 0$ and so for sufficiently large n, we have $b_n^2 \leq |b_n|$ and so $\sum b_n^2$ converges by comparison to $\sum |b_n|$ which is assumed to converge. (This proves that $\ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$ which is a specific case of the more general fact that $L^p(X, \mu) \subseteq L^q(X, \mu)$ whenever $1 \leq p \leq q$ and (X, μ) is a measure space with no sets of arbitrarily small positive measure.)

Problem 18. Let (a_n) be a sequence of positive numbers. The infinite product

$$\prod_{n=1}^{\infty} a_n = a_1 \cdot a_2 \cdot a_3 \cdots$$

is said to converge if there is $L \in (0, \infty)$ such that $\lim_{N\to\infty} \prod_{n=1}^{N} a_n = L$. Otherwise the product is said to diverge to zero or diverge to $+\infty$ if the limit is zero or $+\infty$ respectively. Consider the infinite products

$$\text{(a)} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right), \quad \text{(b)} \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right), \quad \text{(c)} \prod_{n=1}^{\infty} \left(1 - \frac{1}{\log(n)}\right).$$

Show that (a) converges, (b) diverges to $+\infty$ and (c) diverges to 0.

Solution. Let $P = \prod_{n=1}^{\infty} a_n$. Since log is continuous, we can pass it through limits so we see

$$\log(P) = \log\left(\lim_{N\to\infty}\prod_{n=1}^{N} a_n\right) = \lim_{N\to\infty}\log\left(\prod_{n=1}^{N} a_n\right) = \lim_{N\to\infty}\sum_{n=1}^{N}\log(a_n) = \sum_{n=1}^{\infty}\log(a_n).$$

Thus we need only check the sums

$$\text{(a)}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n^2}\right), \quad \text{(b)}\sum_{n=1}^{\infty}\log\left(1+\frac{1}{n}\right), \quad \text{(c)}\sum_{n=1}^{\infty}\log\left(1-\frac{1}{\log(n)}\right).$$

Note that as $x \to 0$, we have $\log(1+x) \sim x$. Thus by the limit comparison test, the first some converges while the second divergest to $+\infty$ and the third diverges to $-\infty$. Undoing the logarithm, this shows that the first product converges, the second diverges to $+\infty$ and the third diverges to 0. [Of course, this is a bit formal; special considerations should be taken if $P = \infty$ or $P = 0$ since $\log(P)$ is not defined in those cases, but it's the same general idea.]

Problem 19. Suppose that $(x(t), y(t))$ for $t \in [a, b]$ is the parameterization a curve and that $x'(t) \neq 0$ for all $t \in [a, b]$. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ as functions of t.

Solution. From the chain rule, we have $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$. Since $x'(t) \neq 0$, this shows that

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

Now

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x'(t)}\frac{d}{dt}\left(\frac{y'(t)}{x'(t)}\right) = \frac{y''(t)x'(t) - y'(t)x''(t)}{x'(t)^3}.$$

Problem 20. Does the series

$$\frac{1}{3} + \frac{1}{3\sqrt{3}} + \frac{1}{3\sqrt{3}\sqrt[3]{3}} + \cdots + \frac{1}{3\sqrt{3}\sqrt[3]{3} + \cdots + \sqrt[n]{3}} + \cdots$$

converge or diverge?

Solution. Put $H_n = \sum_{k=1}^{n} \frac{1}{k}$. Then the series can be written

$$\sum_{n=1}^{\infty} \frac{1}{3^{H_n}}.$$

Now

$$H_n = \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}\int_k^{k+1}\frac{1}{k}dx = \sum_{k=1}^{n}\int_k^{k+1}\frac{1}{\lfloor x\rfloor}dx = \int_1^{n+1}\frac{dx}{\lfloor x\rfloor} \geq \int_1^{n+1}\frac{dx}{x} = \log(n+1).$$

Thus

$$\sum_{n=1}^{\infty} \frac{1}{3^{H_n}} \le \sum_{n=1}^{\infty} \frac{1}{3^{\log(n+1)}} = \sum_{n=1}^{\infty} \frac{1}{e^{\log(3)\log(n+1)}} = \sum_{n=1}^{\infty} \frac{1}{(n+1)^{\log(3)}} < \infty$$

since $\log(3) > 1$.

Problem 21. Evaluate the following limits or prove that they diverge:

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1^2}} + \frac{1}{\sqrt{n^2+2^2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right); \tag{1}$$

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n}} \right); \tag{2}$$

$$\lim_{n\to\infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \cdots + \frac{1}{\sqrt{n^2+n^2}} \right). \tag{3}$$

Solution. For (1), we see

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k^2}} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \frac{1}{\sqrt{1+\left(\frac{k}{n}\right)^2}} = \int_0^1 \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(1).$$

[You can evaluate the integral using the substitution $x = \sinh(t)$ and the identity $\cosh^2(t) - \sinh^2(t) = 1$.]

For (2), call the limit L_2. We have

$$L_2 = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \le \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} = 1.$$

Also

$$L_2 = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}} \ge \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+n}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1.$$

Thus $L_2 = 1$.

Limit (3) diverges. To prove this, we use the same lower bound as in (2), but there are more terms:

$$\sum_{k=1}^{n^2} \frac{1}{\sqrt{n^2+k}} \ge \sum_{k=1}^{n^2} \frac{1}{\sqrt{2n^2}} = \frac{n}{\sqrt{2}} \to \infty.$$

Problem 22. Compute the integral $\displaystyle\int_0^1 \frac{\log(1+t)}{1+t^2} dt$.

Solution. Call the integral I. Using the substitution $t = \tan(\theta)$, we have

$$\begin{aligned} I &= \int_0^{\pi/4} \log(1+\tan(\theta)) d\theta \\ &= \int^{\pi/4} \log(\sec(\theta)(\cos(\theta)+\sin(\theta))\, d\theta \\ &= \int_0^{\pi/4} \log(\cos(\theta)+\sin(\theta))\, d\theta - \int_0^{\pi/4} \log(\cos(\theta))\, d\theta. \end{aligned}$$

But $\cos(\theta) + \sin(\theta) = \sqrt{2}\cos(\pi/4 - \theta)$. Thus

$$\begin{aligned} I &= \int_0^{\pi/4} \log(\sqrt{2}\cos(\pi/4-\theta))\,d\theta - \int_0^{\pi/4} \log(\cos(\theta))\,d\theta \\ &= \int_0^{\pi/4} \frac{\log 2}{2} + \int_0^{\pi/4} \log(\cos(\pi/4-\theta))\,d\theta - \int_0^{\pi/4} \log(\cos(\theta))\,d\theta \\ &= \frac{\pi \log 2}{8} + \int_0^{\pi/4} \log(\cos(\theta))\,d\theta - \int_0^{\pi/4} \log(\cos(\theta))\,d\theta = \frac{\pi\log 2}{8}, \end{aligned}$$

using the substitution $\phi = \pi/4 - \theta$.

Problem 23. Decide whether the following integral converges: $\displaystyle\int_0^\infty \frac{dx}{1 + x^4 \sin^2(x)}$

Solution. The integral converges. Call the integral I and break it up into intervals of length π (since $\sin^2(x)$ is π-periodic):

$$I = \sum_{n=0}^\infty \int_{n\pi}^{(n+1)\pi} \frac{dx}{1 + x^4\sin^2(x)} = \sum_{n=0}^\infty \int_0^\pi \frac{dy}{1 + (y+n\pi)^4 \sin^2(y)} \le \sum_{n=0}^\infty \int_0^\pi \frac{dy}{1 + (n\pi)^4\sin^2(y)}.$$

But since $\sin^2(y)$ is symmetric about $\pi/2$, we have

$$I \le 2\sum_{n=0}^\infty \int_0^{\pi/2} \frac{dy}{1 + (n\pi)^4 \sin^2(y)}.$$

And finally, using $\sin(y) \ge y/2$ for $y \in [0, \pi/2]$, we see

$$I \le \sum_{n=0}^\infty \int_0^{\pi/2} \frac{dy}{1 + \frac{1}{4}(n\pi)^4 y^2} = \frac{\pi}{2} + \sum_{n=1}^\infty \frac{2}{(n\pi)^2} \int_0^{n^2\pi^3/4} \frac{dt}{1+t^2} \le \frac{\pi}{2} + C\sum_{n=1}^\infty \frac{1}{n^2}$$

where $C = \frac{2}{\pi^2}\int_0^\infty \frac{dt}{1+t^2} = \frac{1}{\pi}$. Thus the integral converges since $\sum_{n=1}^\infty \frac{1}{n^2}$ converges.