Qualifying Exam in Basic Analysis, Winter 2014
Duke University, Mathematics Department
Time Allowed: 3 hours

All answers and statements should be proved; no partial credit is given to answers without justification.

Part I: 6 points each, do all 6 questions

1. State and prove the integral test for convergence of series. You may assume the comparison test.
2. Let f be a continuous function from $[0,1]$ to $[0,\infty)$. If $\int_0^1 f(t)dt = 0$, prove that $f(x) = 0$, $\forall x \in [0,1]$.
3. Let f be a bounded Riemann integrable function on $[0,1]$. Prove that f^2 is Riemann integrable on $[0,1]$.
4. Equip $\mathbb{R}^n$ with 2 metrics ρ_1 and ρ_2, where $\rho_1(x,y) = \max_{1\le j\le n} |x^j - y^j|$ and $\rho_2(x,y) = \sqrt{\sum_{j=1}^n |x^j - y^j|^2}$. Prove a set $U \subset \mathbb{R}^n$ is open in $(\mathbb{R}^n, \rho_1)$ if and only if it is open in $(\mathbb{R}^n, \rho_2)$.
5. Let $x_n := \frac{(n^2+2\sin^2(n)+34)\sin(n^3)}{n^2+n+5}$. Prove $\{x_n\}_{n=1}^\infty$ has a convergent subsequence.
6. Let $\{a_n\}_{n=1}^\infty$ be a sequence of positive numbers such that $\sum_{n=1}^\infty a_n$ converges. Prove $\sum_{n=1}^\infty a_n^2$ converges.

Part II: 10 points each. Do all 6 questions.

1. Let $g(x)$ be a positive Riemann integrable function. Show there exists $c \in [1,2]$ so that
$$\int_1^2 e^{t^2} g(t)dt = e^{c^2} \int g(t)dt.$$
2. Let $f(x) := \sum_{k=0}^\infty \frac{cos(kx)}{k^3+k^2+1}$. Prove that f is differentiable with derivative $\sum_{k=0}^\infty \frac{-k\sin(kx)}{k^3+k^2+1}$.
3. Let $U \subset \mathbb{R}^p$ be open. Let $f : U \to \mathbb{R}^n$ be a C^1 function. Suppose that for some $x_0 \in U$, Df_{x_0} is rank p. Show that there is a change of variables in some neighborhood of $f(x_0)$ so that in the new coordinate system, f is a linear inclusion.
4. Show there exists a $f \in C([0,1],\mathbb{R})$ solving the equation
$$f(x) = \frac{1}{1066}\int_0^x e^{-s^2x^2} f(s)^2 ds + \sin(2013x).$$
5. Use Taylor's theorem to prove that $\sum_{k=1}^\infty \ln(1+k^{-2/3})$ diverges but $\sum_{k=1}^\infty [\ln(1+k^{-2/3}) - k^{-2/3}]$ converges. You may use standard convergence tests.
6. Prove a linear function $L : V_1 \to V_2$ between two normed vector spaces is continuous if and only if there exists $c > 0$ such that $|L(v)| \le c|v|, \forall v \in V_1$.