ANALYSIS NOTES

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REAL ANALYSIS

§0: Some basics

Liminfs and Limsups

• **Def.-** Let $(x_n) \subseteq \mathbb{R}$ be a sequence. The **limit inferior** of (x_n) is defined by

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m$$

and, similarly, the **limit superior** of (x_n) is

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m.$$

• **Remark.-**

$$\liminf_{n\to\infty} x_n = \sup_n \inf_{m\geq n} x_m$$

$$\limsup_{n\to\infty} x_n = \inf_n \sup_{m\geq n} x_m.$$

• **Def.-** A number $\xi \in \mathbb{R} \cup \{-\infty, +\infty\}$ is a **subsequential limit** of (x_n) if there exists a subsequence of x_n that converges to ξ. We denote by E the set of subsequential limits of (x_n). • **Lemma.-**

$$\liminf(x_n) = \inf E$$

$$\limsup(x_n) = \sup E.$$

In fact, E is closed so we can replace the above with min and max.

• **Lemma.-** For any sequence (x_n), $\liminf(x_n) \leq \limsup(x_n)$ and (x_n) converges to L if and only if $\liminf(x_n) = L = \limsup(x_n)$.

§1: Measure Theory [SS]

Preliminaries

• **Thm.-** Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as a disjoint union of countably many open intervals.

• **Thm.-** Every open subset $\mathcal{O}$ of $\mathbb{R}^d$ can be written as a countable union of almost disjoint closed cubes.

The exterior measure

• **Def.-** If $E \subseteq \mathbb{R}^d$ is any subset, the **exterior measure of** E is

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : \bigcup_{n=1}^{\infty} Q_j \supseteq E, Q_j \text{ cubes} \right\}.$$

• **Prop.-** (Properties of the exterior measure)
(i) If $E_1 \subseteq E_2$ then $m_*(E_1) \leq m_*(E_2)$.
(ii) If $E = \cup_{j=1}^{\infty} E_j$ then $m_*(E) \leq \sum_j m_*(E_j)$.
(iii) If $E \subseteq \mathbb{R}^d$ then $m_*(E) = \inf m_*(\mathcal{O})$ where the infimum is taken over all open $\mathcal{O}$ that contain E.
(iv) If $E = E_1 \cup E_2$ and $d(E_1, E_2) > 0$ then $m_*(E) = m_*(E_1) + m_*(E_2)$.
(v) If E is an almost disjoint union of countably many cubes Q_j then $m_*(E) = \sum_j m_*(Q_j)$.

Measurable sets and Lebesgue measure

• **Def.-** A subset E of $\mathbb{R}^d$ is **measurable** if for all $\epsilon > 0$ there exists an open $\mathcal{O} \subseteq \mathbb{R}^d$ with $E \subseteq \mathcal{O}$ and $m_*(\mathcal{O} \setminus E) \leq \epsilon$.

• **Prop.-** (Properties of measurable sets)
(i) Every open set of $\mathbb{R}^d$ is measurable.
(ii) If $m_*(E) = 0$ then E is measurable – thus if $F \subseteq E$ and $m(E) = 0$ then F is measurable.
(iii) A countable union of measurable sets is measurable.
(iv) Closed sets are measurable.
(v) The complement of a measurable set is measurable.
(vi) A countable intersection of measurable sets is measurable.

• **Thm.-** If $E = \cup_{j=1}^{\infty} E_j$ is a countable union of disjoint measurable sets then $m(E) = \sum_j m(E_j)$. **Cor.-** Suppose $E_1, E_2, \ldots$ are measurable. If $E_j \nearrow E$ then $\lim_{j\to\infty} m(E_j) = m(E)$. If $E_j \searrow E$ and $m(E_j) < \infty$ for some j then $\lim_{j\to\infty} m(E_j) = m(E)$.

• **Thm.-** Suppose $E \subseteq \mathbb{R}^d$ is measurable. Then for every $\epsilon > 0$:
(i) There exists an open $\mathcal{O}$ with $E \subseteq \mathcal{O}$ and $m(\mathcal{O} - E) \leq \epsilon$.
(ii) There exists a closed F with $F \subseteq E$ and $m(E - F) \leq \epsilon$.
(iii) Furthermore, if $m(E) < \infty$ then the F in (ii) can be taken to be compact.
(iv) If $m(E) < \infty$ then there exists a finite union $F = \cup_{j=1}^{N} Q_j$ of closed cubes with $m(E \triangle F) \leq \epsilon$.

Borel subsets

• **Def.-** The **Borel σ-algebra** of $\mathbb{R}^d$, denoted $\mathcal{B}(\mathbb{R}^d)$, is the σ-algebra generated by the open subsets of $\mathbb{R}^d$. A **G-delta** (or G_δ) set is a countable intersection of open sets. An **F-sigma** (or F_σ) set is a countable union of closed sets.

• **Thm.-** Let $E \subseteq \mathbb{R}^d$ be any subset. The following are equivalent.
(i) E is Lebesgue measurable.
(ii) E differs from a G_δ by a set of measure 0.
(iii) E differs from an F_σ by a set of measure 0.

• **Cor.-** The Lebesgue σ-algebra is the completion of the Borel σ-algebra.

Measurable functions

• **Remark.-** We allow functions to take the values $-\infty$ and ∞.

• **Def.-** A function f on $\mathbb{R}^d$ is **measurable** if, for all $a \in \mathbb{R}$, $f^{-1}([-\infty, a))$ is measurable.

• **Lemma.-** The following are equivalent for a function f.
(i) f is measurable. (ii) $f^{-1}(\mathcal{O})$ is measurable for every open $\mathcal{O} \subseteq \mathbb{R}$.
(iii) $f^{-1}(F)$ is measurable for every closed $F \subseteq \mathbb{R}$.

• **Prop.-** (Properties of measurable functions)
(i) If f is measurable on $\mathbb{R}^d$ and finite-valued and $\Phi : \mathbb{R} \to \mathbb{R}$ is continuous then $\Phi \circ f$ is measurable.
(ii) If $\{f_k\}$ is a sequence of measurable functions then the pointwise sup, inf, limsup, and liminf are all measurable. Then pointwise limit, when it exists – at least a.e. – is also measurable.
(iii) If f is measurable then f^k $(k \geq 1)$ is also measurable.
(iv) If f and g are measurable and finite-valued then $f+g$ and fg are also measurable. (iv) If f is measurable and $f(x) = g(x)$ for a.e. x then g is measurable.

Approximations by simple functions

• **Thm.-** If f is a *non-negative* measurable function then there exists an increasing sequence of non-negative simple functions $\{\phi_n\}$ that converges pointwise to f.

• **Thm.-** If f is *any* measurable function then there exists a sequence of simple functions $\{\phi_k\}_{k=1}^{\infty}$ with $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ that converges pointwise to f.

• **Thm.-** If f is *any* measurable function then there exists a sequence of *step* functions (simple functions made with rectangles only) that converges pointwise to f *almost everywhere*.

Littlewood's three principles

• **Remark.-** Littlewood's three principles are:
(i) Every set is nearly a finite union of intervals. (c.f. some theorem from before)
(ii) Every function is nearly continuous. (c.f. Lusin's theorem)
(iii) Every convergent sequence is nearly uniformly convergent. (c.f. Egorov's theorem)

Thm.- (Egorov) Let $\{f_k\}$ be a sequence of measurable functions *supported on E where $m(E) < \infty$* such that $f_k(x) \to f(x)$ for a.e. x. Then for every $\epsilon > 0$ there exists a *closed* subset $A_\epsilon \subseteq E$ with $m(E - A_\epsilon) \leq \epsilon$ such that $f_k \to f$ uniformly on A_ϵ. **Example.-** The convergence of $f_n(x) = x^n$ on $[0, 1]$.

Thm.- (Lusin) Suppose f is a measurable, *finite-valued* function on E where E *is of finite measure*. Then for every $\epsilon > 0$ there exists a *closed* $F_\epsilon \subseteq E$ with $m(E - F_\epsilon) \leq \epsilon$ wuch that $f|_{F_\epsilon}$ is continuous. **Remark.-** This is different than saying f is continuous on F_ϵ, e.g. $\chi_{\mathbb{Q}}$ on $[0, 1]$.

§2: Integration Theory [SS]

The Lebesgue Integral

• **Def.-** Given a simple function $\phi = \sum_k c_k \chi_{E_k}$ we define its **integral** to be

$$\int \phi := \sum_k c_k m(E_k).$$

Fact.- This is independent of the representation.

• **Def.-** Now given a function f that is (i) bounded and (ii) supported on a set E of finite measure, and given a sequence $\{\phi_n\}$ of simple functions (i) bounded (uniformly) by some M (ii) supported on the set E with (iii) $\phi_n(x) \to f(x)$ for a.e. x then we define the **Lebesgue integral** of f by

$$\int f := \lim_{n\to\infty} \int \phi_n.$$

Fact.- The limit always exists, and it does not depend on the sequence ϕ_n. If f is measurable then such a sequence always exists.

• **Def.-** If f is a (i) measurable, (ii) non-negative (but we allow infinite values) then its **Lebesgue integral** is given by

$$\int f := \sup_g \int g$$

where the sup is taken over all measurable bounded g that are supported on a set of finite measure. We say such a function is **Lebesgue integrable** if the integral is finite.

• **Def.-** Now given a measurable function f such that (i) $|f|$ is integrable then we define its **Lebesgue integral** by

$$\int f := \int f_+ - \int f_-.$$

• **Fact.-** All the definitions above agree.

• **Prop.-** The Lebesgue and Riemann integrals agree for Riemann-integrable functions defined on closed intervals.

• **Prop.-** The integral of Lebesgue integrable functions is linear, additive, monotonic and satisfies the triangle inequality.

• **Prop.-** Suppose f is integrable on $\mathbb{R}^d$. Then for every $\epsilon > 0$
(i) There exists a ball $B \subseteq \mathbb{R}^d$ such that $\int_{B^c} |f| < \epsilon$.
(ii) There exists a $\delta > 0$ such that $\int_E |f| < \epsilon$ whenever $m(E) < \delta$.

• **Lemma.-** (Fatou) Suppose $\{f_n\}$ is a sequence of non-negative measurable functions. If $\lim_{n\to\infty} f_n(x) = f(x)$ for a.e. x then

$$\int f \leq \liminf_{n\to\infty} \int f_n.$$

• **Cor.-** Suppose f is a non-negative measurable function and $\{f_n\}$ is a sequence of non-negative measurable functions with $f_n(x) \leq f(x)$ and $f_n(x) \to f(x)$ for a.e. x. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

• **Cor.-** (Monotone Convergence Theorem) Suppose $\{f_n\}$ is a sequence of *non-negative* measurable function with $f_n \nearrow f$. Then

$$\lim_{n\to\infty} \int f_n = \int f.$$

• **Thm.-** (Dominated Convergence Theorem) Suppose $\{f_n\}$ is a sequence of measurable functions such that $f_n(x) \to f(x)$ a.e. x. If $|f_n(x)| \leq g(x)$, where $g(x)$ is integrable then

$$\int |f_n - f| \to 0 \quad \text{as } n \to \infty,$$

and thus

$$\int f_n \to \int f \quad \text{as } n \to \infty.$$

• **Counterexample.-** Consider the function $f_n(x) = 1/n$. Then $f_n \to f$ where $f = 0$. This provides a counterexample for Dominated Convergence when the f_n are not dominated by an L^1 function, and also a counterexample to monotone convergence when $f_n \searrow f$ – and thus $-f_n \nearrow -f$, i.e. negative functions.

• **Prop.-** (Tchebyshev's inequality) Let f be *integrable*. Then for all $\alpha > 0$

$$m(\{x : |f(x)| > \alpha\}) \leq \frac{\|f\|_1}{\alpha}.$$

The space L^1

• **Def.-** The space $L^1 = L^1(\mathbb{R}^d)$ is the space of equivalence classes of Lebesgue integrable functions, where we regard two functions as equivalent if the are equal almost everywhere. **Remark.-** The integral is still defined as an operator in L^1 and $\|f\| = \int |f|$ defines a norm on L^1, and thus $d(f, g) = \int |f - g|$ defines a metric on L^1.

• **Thm.-** (Riesz-Fischer) The vector space L^1 is complete in its metric. Moreover, any Cauchy sequence $\{f_n\}$ in L^1 has a subsequence that converges pointwise almost-everywhere.

• **Thm.-** The following families are dense in L^1.
(i) Simple functions.
(ii) Step functions (characteristic functions of finite union of rectangles).
(iii) Continuous functions with compact support.

• **Thm.-** Let $f(x)$ be integrable, $h \in \mathbb{R}$, $\delta \in \mathbb{R}_{>0}$. Then $f(x - h)$, $f(\delta x)$, $f(-x)$ are integrable and
(i) $\int f(x - h) = \int f(x)$.
(ii) $\int f(\delta x) = \delta^d \int f(x)$.
(iii) $\int f(-x) = \int f(x)$.

Fubini's Theorem

• **Def.-** For this section we set $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, and thus a point on $\mathbb{R}^d$ takes the form (x, y) where $x \in \mathbb{R}^{d_1}$, $y \in R^{d_2}$. If $f(x, y)$ is a function on $\mathbb{R}^d$ we define the **slice** $f^y(x) := f(x, y)$, which is then a function on $\mathbb{R}^{d_1}$, and $f_x(y)$ similarly. Given a set $E \subseteq \mathbb{R}^d$ denote its **slice** by $E^y = \{x \in \mathbb{R}^{d_1} : (x, y) \in E\}$ and $E_x = \{y \in \mathbb{R}^{d_2} : (x, y) \in E\}$.

• **Thm.-** (Fubini) Suppose $f(x, y)$ is *integrable* on $\mathbb{R}^d$. Then, for almost every $y \in \mathbb{R}^{d_2}$,
(i) The slice f^y is integrable on $\mathbb{R}^{d_1}$.
(ii) The function $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$.
(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)dx \right) dy = \int_{\mathbb{R}^d} f d(x, y).$$

• **Thm.-** (Tonelli) Suppose $f(x,y)$ is a *non-negative* measurable function on $\mathbb{R}^d$. Then, for almost every $y \in \mathbb{R}^{d_2}$,
(i) The slice f^y is measurable on $\mathbb{R}^{d_1}$.
(ii) The function $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is measurable on $\mathbb{R}^{d_2}$.
(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^d} f(x,y)d(x,y)$$

(where now this can be an equality $\infty = \infty$).

• **Cor.-** If E is a measurable set in $\mathbb{R}^d$ then, for almost all $y \in \mathbb{R}^{d_2}$, the slice

$$E^y = \{x \in \mathbb{R}^{d_1} : (x,y) \in E\}$$

is a measurable subset of $\mathbb{R}^{d_1}$. Moreover, $m(E^y)$ is a measurable function of $\mathbb{R}^{d_2}$ and $m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)dy$.

• **Prop.-** If $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$ and $m_*(E_2) > 0$ then E_1 is measurable.

• **Lemma.-** If $E_1 \subseteq \mathbb{R}^{d_1}$, $E_2 \subseteq \mathbb{R}^{d_2}$ are any sets, then

$$m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2).$$

(with the understanding that $0 \cdot \infty = 0$).

• **Prop.-** If E_1, E_2 are measurable subsets of $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$ resp. then $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$ and, moreover,

$$m(E) = m(E_1)m(E_2).$$

(where $0 \cdot \infty = 0$). **Cor.-** If $f(x)$ is any function on $\mathbb{R}^{d_1}$ then $\widetilde{f}(x,y) := f(x)$ is measurable in $\mathbb{R}^d$. **Cor.-** Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$ and let

$$A := \{(x,y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Then (i) A is measurable in $\mathbb{R}^{d+1}$ if and only if f is measurable on $\mathbb{R}^d$ and, whenever these hold, (ii) $m(A) = \int f(x)$.

Convolutions

• If f, g are measurable functions on $\mathbb{R}^d$ then $f(x-y)g(y)$ is measurable in $\mathbb{R}^{2d}$.

• If, furthermore, f, g are integrable on $\mathbb{R}^d$ then $f(x-y)g(y)$ is integrable in $\mathbb{R}^{2d}$.

• The **convolution** of f and g is

$$(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)g(y)dy.$$

• The function $(f * g)(x)$ is well-defined for almost all x.

• $f * g$ is integrable whenever f and g are, and

$$\|f * g\|_{L^1} \leq \|f\|_{L^1} \|g\|_{L^1}$$

with equality whenever f and g are non-negative.

§3: Differentiation and Integration [SS]

Differentiation of the integral.

• **Notation.-** Throughout this section, B always denotes *balls*.

• **Remark.-** For a *continuous* function f on $\mathbb{R}^d$ we have

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)}\int_B f(y)dy = f(x).$$

• **Def.-** Given an *integrable* function f on $\mathbb{R}^d$ we define its **Hardy-Littlewood maximal function** f^* by

$$f^*(x) = \sup_{B\ni x} \frac{1}{m(B)}\int_B |f(y)|dy.$$

• **Thm.-** (Maximal theorem) Suppose f is integrable. Then:
(i) f^* is measurable.
(ii) $f^*(x) < \infty$ for a.e. x.
(iii) $f^*(x)$ satisfies

$$m(\{x\in\mathbb{R}^d : f^*(x) > \alpha\}) \le \frac{A}{\alpha}\|f\|_{L^1}$$

for all $\alpha > 0$, where $A = 3^d$. **Proof idea.-** (i) is easy, (ii) follows from (iii). (iii) is hard and uses a version of Vitali covering argument. **Remark.-** (iii) is a weak-type inequality, i.e. weaker than inequality on L^1-norms, by Tchebyshev's inequality. Observe we could have defined $f^*(x)$ using balls centered at x. Then this inequality still holds with the same $A = 3^d$ – see Folland. **Remark.-** The function $f^*(x)$ may not be L^1 – see $f^*(x)$ when $f(x) = \chi_{[0,1]}$.

• **Thm.-** (Lebesgue differentiation theorem) If f is *integrable* then

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)}\int_B f(y)dy = f(x) \quad \text{for a.e. } x.$$

Cor.- $f^*(x) \ge |f(x)|$ for a.e. x.

• **Def.-** A measurable function f is **locally integrable** if, for all balls B, $f\chi_B$ is integrable. We denote by $L^1_{loc}(\mathbb{R}^d)$ the space of locally integrable functions. **Remark.-** The Lebesgue differentiation theorem holds for locally integrable functions.

• **Def.-** If E is a measurable set and $x\in\mathbb{R}^d$ we say x is a point of **Lebesgue density** of E if

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{m(B\cap E)}{m(B)} = 1.$$

• **Cor.-** (Lebesgue's density theorem) Suppose E is a measurable subset of $\mathbb{R}^d$. Then:
(i) Almost every $x\in E$ is a point of Lebesgue density of E.
(ii) Almost every $x\notin E$ is not a point of Lebesgue density of E – and, in fact, the limit above is 0 for almost all $x\notin E$.

• **Def.-** If f is locally integrable on $\mathbb{R}^d$ the **Lebesgue set** of f consists of all points $x\in\mathbb{R}^d$ for which $f(x) < \infty$ and

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)}\int_B |f(y)-f(x)|dy = 0.$$

Remark.- If f is continuous at x then x is in the Lebesgue set of x. If x is in the Lebesgue set of x then

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)} \int_B f(y)dy = f(x).$$

• **Remark.-** The Lebesgue set of f depends on the choice of representative.

• **Cor.-** If f is locally integrable on $\mathbb{R}^d$ then almost every point belongs to the Lebesgue set of f.

• **Def.-** A collection of sets $\{U_\alpha\}$ is said to **shrink regularly** to x, or to have **bounded eccentricity** at x, if there is a constant $c > 0$ such that for each U_α there is a ball B with $x \in B$, $U_\alpha \subseteq B$ and $m(U_\alpha) \geq cm(B)$. (Perhaps we also need that x is contained in arbitrarily small U_α's? Folland has a more clear discussion).

• **Cor.-** Suppose f is locally integrable on $\mathbb{R}^d$. If $\{U_\alpha\}$ shrinks regularly to x then

$$\lim_{\substack{m(U_\alpha)\to 0\\ U_\alpha\ni x}} \frac{1}{m(U_\alpha)} \int_{U_\alpha} f(y)dy = f(x)$$

for all x in the Lebesgue set of f – and thus for almost every x.

Approximations to the identity

Def.- A family $\{K_\delta\}_{\delta>0}$ of *integrable* functions on $\mathbb{R}^d$ are an **approximation to the identity** if:
(i) $\int K_\delta(x)dx = 1$.
(ii) $|K_\delta(x)| \leq A\delta^{-d}$.
(iii) $|K_\delta(x)| \leq A\delta/|x|^{d+1}$.
for all $\delta > 0$ and $x \in \mathbb{R}^d$, where A is a constant independent of δ.

Thm.- If $\{K_\delta\}$ is an approximation to the identity and f is *integrable* on $\mathbb{R}^d$ then

$$(f * K_\delta)(x) \to f(x) \quad \text{as} \quad \delta \to 0$$

whenever x is in the Lebesgue set of f – and thus for a.e. x.

Thm.- With the hypotheses of the previous theorem, we also have

$$\|(f * K_\delta) - f\|_{L^1} \to 0 \quad \text{as} \quad \delta \to 0.$$

Remark.- Recall $f * K_\delta$ are integrable.

Differentiability of functions

• **Def.-** Let γ be a parametrized curve in the plane given by $z(t) = (x(t), y(t))$ where $a \leq t \leq b$ and $x(t), y(t)$ are continuous real valued functions on $[a, b]$. Then γ is **rectifiable** if there exists some $M > 0$ such that, for any partition $a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$,

$$\sum_{j=1}^{N} |z(t_j) - z(t_{j-1})| \leq M.$$

The **length** $L(\gamma)$ of γ is the supremum over all partitions of the left-hand side – or, equivalently, the infimum of all M that satisfy the above.

• **Def.-** Similarly, if $F : [a,b] \to \mathbb{C}$ is continuous and $a = t_0 < t_1 < \cdots < t_N = b$ then the **variation** with respect to this partition is given by

$$\sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|.$$

The function F is said to be of **bounded variation** if there exists some uniform bound for all variations.

• **Thm.-** A real-valued function F on $[a,b]$ is of bounded variation if and only if F is the difference of two increasing (not necessarily strictly) bounded functions.

• **Thm.-** If F is of bounded variation on $[a,b]$ then F is differentiable almost everywhere. **Cor.-** If F is increasing and continuous then F' exists almost everywhere. Moreover, F' is measurable, non-negative and

$$\int_a^b F'(x)dx \leq F(b) - F(a).$$

In particular, if F is bounded then F' is integrable. **Remark.-** There is a continuous function for which the left-hand side is 0 and the right-hand side is 1, called the Cantor function.

• **Def.-** A function F on $[a,b]$ is **absolutely continuous** of, for any $\epsilon > 0$, there exists some $\delta > 0$ such that

$$\sum_{k=1}^{N} |F(b_k) - F(a_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=1}^{N} (b_k - a_k) < \delta$$

where the (a_k, b_k), $k = 1, \ldots, N$, are disjoint intervals. **Remark.-** Absolute continuity implies uniform (and thus plain-old) continuity. It also implies bounded variation. The total variation is then also absolutely continuous and thus F is the difference of two *continuous* monotonic functions. If $F(x) = \int_a^x f(y)dy$, where f is integrable, then F is absolutely continuous.

• **Thm.-** If F is absolutely continuous on $[a,b]$ then F' exists almost everywhere and it is integrable. Moreover,

$$F(b) - F(a) = \int_a^b F'(y)dy.$$

Conversely, if f is integrable on $[a,b]$ there exists an absolutely continuous function F such that $F' = f$ almost everywhere and, in fact, we may take $F(x) = \int_a^x f(y)dy$.

• **Thm.-** If F is a bounded increasing function on $[a,b]$ then F' exists almost everywhere.

§4: Abstract Measure and Integration Theory [SS]

Abstract measure spaces

• **Def.-** Let X be a non-empty set. A **σ-algebra** $\mathcal{M}$ is a non-empty collection of subsets of X that is closed under complements and countable unions. **Remark.-** A σ-algebra $\mathcal{M}$ is then closed under countable intersection as well. Moreover, $X, \phi \in \mathcal{M}$.

• **Def.-** Let $\mathcal{M}$ be a σ-algebra. A **measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \to [0, \infty]$ such that whenever $E_1, E_2, \dots$ is a countable *disjoint* family of sets in $\mathcal{M}$ then

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu(E_n).$$

Remark.- Observe then that $\mu(\phi) = 0$ and thus the above formula holds for finite unions too.

• **Def.-** A **measure space** is a triple $(X, \mathcal{M}, \mu)$ where X is a set, $\mathcal{M}$ is a σ-algebra on X and μ is a measure on $\mathcal{M}$. It is said to be **complete** if whenever $F \in \mathcal{M}$ is such that $\mu(F) = 0$ and $E \subseteq F$ then $E \in \mathcal{M}$. It is said to be **σ-finite** whenever X is a countable union of sets of finite measure.

Exterior measures, Carathéodory's theorem

• **Def.-** If X is a non-empty set, an **exterior measure** or **outer measure** μ_* on X is a function from *all* subsets of X to $[0, \infty]$ that satisfies:
(i) $\mu_*(\phi) = 0$.
(ii) If $E_1 \subseteq E_2$ then $\mu_*(E_1) \leq \mu_*(E_2)$.
(iii) If $E_1, E_2, \dots$ is a countable family of sets then

$$\mu_*(\bigcup_{j=1}^{\infty} E_j) \leq \sum_{j=1}^{\infty} \mu_*(E_j).$$

A subset $E \subseteq X$ is then called **Carathéodory measurable**, or **measurable** if for every $A \subseteq X$ we have

$$\mu_*(A) = \mu_*(E \cap A) + \mu_*(E^c \cap A).$$

• **Thm.-** Given an exterior measure μ_* on X the collection $\mathcal{M}$ of measurable subsets is a σ-algebra, and μ_* restricted to $\mathcal{M}$ is a measure. Moreover, the resulting measure space is complete.

Metric exterior measures

• **Def.-** If (X, d) is a metric space the **Borel σ-algebra** $\mathcal{B}_X = \mathcal{B}$ on X is the smallest σ-algebra that contains all open sets of X. An exterior measure μ_* on X is a **metric exterior measure** if

$$\mu_*(A \cap B) = \mu_*(A) + \mu_*(B) \quad \text{whenever} \quad d(A, B) > 0.$$

• **Thm.-** If μ_* is a metric exterior measure on X then the Borel sets in X are measurable – thus, μ_* restricted to $\mathcal{B}_X$ is a measure.

• **Def.-** Given a metric space X, a measure on the Borel sets is called a **Borel measure**.

• **Prop.-** Suppose the Borel measure μ is finite on all balls in X of finite radius. Then for any Borel set E an any $\epsilon > 0$ there is an open set $\mathcal{O}$ and a closed set F with $F \subseteq E \subseteq \mathcal{O}$ such that $\mu(F - E) < \epsilon$, $\mu(\mathcal{O} - E) < \epsilon$.

The extension theorem

• **Def.-** If X is a non-empty set, an **algebra** on X is a non-empty collection $\mathcal{A}$ of subsets closed under complements and *finite* unions – and thus under *finite* intersection. A pre-measure is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ with:
(i) $\mu_0(\phi) = 0$,
(ii) If $E_1, E_2, \ldots$ is a countable disjoint collection of sets in $\mathcal{A}$ with $\cup_{n=1}^{\infty} E_n \in \mathcal{A}$ – e.g. finite union – then

$$\mu_0(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \mu_0(E_n).$$

• **Thm.-** (Carathéodory's Extension Theorem) Suppose $\mathcal{A}$ is an algebra on X and μ_0 is a premeasure on $\mathcal{A}$. Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ that extends μ_0. This extension is unique whenever (X, μ_0) is σ-finite.

Integration on a measure space

• Fix throughout this section a σ*-finite* measure space $(X, \mathcal{M}, \mu)$.

• **Def.-** A function f on X (with values on $\mathbb{R} \cup \{\pm\infty\}$) is **measurable** if for all $a \in \mathbb{R}$ $f^{-1}([-\infty, a))$ is measurable. **Remark.-** If $\{f_n\}$ is a sequence of measurable functions then the pointwise sup, inf, limsup and liminf and lim – when it exists – are measurable. If f, g are measurable and of finite value then $f + g$ and fg are measurable.

• **Def.-** A **simple function** on X is a function of the form $\phi(x) = \sum_{k=1}^{N} a_k \chi_{E_k}$ where $a_k \in \mathbb{R}$ and the E_k are measurable.

• **Thm.-** A measurable function f is the pointwise limit of a sequence $\{\phi_k\}$ of simple functions. Moreover, the ϕ_k may be taken such that $|\phi_k(x)| \leq |\phi_{k+1}(x)|$ for all k. **Remark.-** We use σ-finiteness here, but the following results don't (I think?).

• **Thm.-** (Egorov's) If $\{f_k\}$ is a sequence of measurable functions defined on a measurable set E *of finite measure* and $f_k(x) \to f(x)$ almost everywhere then for each $\epsilon > 0$ there is a measurable set $A_\epsilon \subseteq E$ with $\mu(E - A_\epsilon) \leq \epsilon$ such that $f_k \to f$ uniformly on A_ϵ.

• **Def.-** Given a simple function $\phi = \sum a_k \chi_{E_k}$ on X, $\int_X \phi d\mu = \sum a_k \mu(E_k)$. Given a non-negative function f on X we define

$$\int_X f d\mu := \sup \left\{ \int_X \phi d\mu : 0 \leq \phi \leq f, \phi \text{ simple} \right\}.$$

Finally, given any function f, $\int_X f d\mu := \int_X f_+ d\mu - \int_X f_- d\mu$.

• **Def.-** A measurable function f on X is **integrable** if $\int |f| d\mu < \infty$.

• **Lemma.-** (Fatou) If $\{f_n\}$ is a sequence of *non-negative* measurable functions on X then

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

• **Thm.-** (Monotone convergence) If $\{f_n\}$ is a sequence of *non-negative* measurable functions on X with $f_n \nearrow f$ then

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

• **Thm.-** (Dominated convergence) If $\{f_n\}$ is a sequence of measurable functions with $f_n(x) \to f(x)$ a.e. and such that $|f_n(x)| < g(x)$ for an integrable function g then

$$\int |f - f_n| d\mu \to 0 \quad \text{as} \quad n \to \infty$$

and thus

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

The spaces L^1 and L^2

• **Def.-** The space $L^1(X,\mu)$ is the space of integrable functions modulo functions that vanish everywhere. The space $L^2(X,\mu)$ is the space of square-integrable (usually $\mathbb{C}$-valued) functions modulo functions that vanish everywhere.

• **Thm.-** The space $L^1(X,\mu)$ is a *complete* normed vector space. The space $L^2(X,\mu)$ is a (possible non-separable) Hilbert space.

Product measures and a general Fubini theorem

• In this section, we fix two *complete* and *σ-finite* measure spaces $(X_1, \mathcal{M}_1, \mu_1)$ and $(X_2, \mathcal{M}_2, \mu_2)$.

• **Def.-** A **measurable rectangle**, or rectangle for short, is a subset of $X_1 \times X_2$ of the form $A \times B$ where $A \subseteq X_1$ and $B \subseteq X_2$ are measurable. **Remark.-** The collection $\mathcal{A}$ of sets in X that are finite unions of disjoint rectangles is an algebra of subsets of X.

• **Prop.-** There is a unique pre-measure μ_0 on $\mathcal{A}$ such that $\mu_0(A \times B) = \mu_1(A)\mu_2(B)$ for all rectangles $A \times B$.

• **Def.-** Let $\mathcal{M}$ be the σ-algebra generated by $\mathcal{A}$. Then μ_0 extends to a measure $\mu_1 \times \mu_2$ on $\mathcal{M}$. Given E in $\mathcal{M}$, $x_1 \in X_1$ and $x_2 \in X_2$ the **slices** are defined by $E_{x_1} := \{x_2 \in X_2 : (x_1, x_2) \in E\}$ and $E^{x_2} := \{x_1 \in X_1 : (x_1, x_2) \in E\}$.

• **Prop.-** If E is measurable in $X_1 \times X_2$ then E^{x_2} is μ_1-measurable for a.e. $x_2 \in X_2$. The function $\mu_1(E^{x_2})$ is μ_2-measurable and

$$\int_{X_2} \mu_1(E^{x_2}) d\mu_2 = (\mu_1 \times \mu_2)(E).$$

Remark.- Of course, a similar statement holds after replacing X_1 with X_2.

• **Thm.-** (Generalized Fubini) In the above setting, suppose $f(x_1, x_2)$ is *integrable* on $(X_1 \times X_2, \mu_1 \times \mu_2)$. Then:
(i) For a.e. $x_2 \in X_2$ the function $f(x_1, x_2)$ is μ_1-integrable (in particular, measurable).
(ii) The function $\int_{X_1} f(x_1, x_2) d\mu_1$ is μ_2-integrable.
(iii)

$$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \times \mu_2).$$

• **Thm.-** (Generalized Tonelli) Again in the above setting, if $f(x_1, x_2)$ is *non-negative* and measurable on $(X_1 \times X_2, \mu_1 \times \mu_2)$ then:
(i) For a.e. $x_2 \in X_2$ the function $f(x_1, x_2)$ is μ_1-measurable.
(ii) The function $\int_{X_1} f(x_1, x_2) d\mu_1$ is μ_2-measurable.
(iii)

$$\int_{X_2} \left(\int_{X_1} f(x_1, x_2) d\mu_1 \right) d\mu_2 = \int_{X_1 \times X_2} f(x_1, x_2) d(\mu_1 \times \mu_2).$$

§5: L^p spaces [F]

Basic Theory

• Fix a measure space $(X, \mathcal{M}, \mu)$. On this section we consider *complex-valued* functions.

• **Def.-** If f is a measurable function on X and $0 < p < \infty$ then define its **p-norm** to be

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{1/p}.$$

Define the space $L^p(X, \mathcal{M}, \mu)$ to be the set of measurable functions f with $\|f\|_p < \infty$ – modulo almost everywhere equality. **Remark.-** L^p is indeed a vector space.

• **Lemma.-** (Hölder's inequality) Suppose $1 < p < \infty$ and $p^{-1} + q^{-1} = 1$ – we say p and q are **Hölder conjugates**. If f and g are measurable functions on X then

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

In particular, if $f \in L^p$ and $g \in L^q$ then $fg \in L^1$. Moreover, equality holds precisely when $\alpha|f|^p = \beta|g|^q$ for some α, β not both zero.

• **Thm.-** (Minkowsky's Inequality) If $1 \leq p < \infty$ and $f, g \in L^p$ then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$. **Cor.-** For $1 \leq p < \infty$, L^p is a normed vector space.

• **Thm.-** For $1 \leq p < \infty$, L^p is a Banach space – i.e. it is complete.

• **Prop.-** For $1 \leq p < \infty$, the set of simple functions with support of finite-measure is dense in L^p.

The case $p = \infty$

Def.- If f is measurable on X we define its **L^∞-norm** by

$$\|f\|_\infty := \inf \{a \geq 0 L \mu(\{x : |f(x)| > a\} = 0)\}$$

(with the convention $\inf = \infty$). We define $L^\infty(X, \mathcal{M}, \mu)$ to be the space of measurable functions $f : X \to \mathbb{C}$ with $\|f\|_\infty < \infty$ – modulo everywhere equivalence. **Remark.-** f is in L^∞ if and only if there is a bounded measurable function g with $f = g$ a.e.

Thm.-
(i) If f and g are measurable functions on X then $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$ – extension of Hölder's inequality.
(ii) $\| \cdot \|_\infty$ is a norm on L^∞.
(iii) $\|f_n \to f\|_\infty \to 0$ if and only $f_n \to f$ uniformly outside a set of measure zero.
(iv) L^∞ is a Banach space. (v) The simple functions are dense in L^∞.

Relations between L^p-spaces

Prop.- If $0 < p < q < r \leq \infty$ then $L^q \subseteq L^p + L^r$; that is, each $f \in L^q$ is the sum of a function in L^p and a function in L^r.

Prop.- If $0 < p < q < r \leq \infty$ then $L^p \cap L^r \subseteq L^q$, with

$$\|f\|_q \leq \|f\|_p^\lambda \|f\|_r^{1-\lambda}$$

where $\lambda \in (0, 1)$ is such that $q^{-1} = \lambda p^{-1} + (1 - \lambda) r^{-1}$.

COMPLEX ANALYSIS

§2: Cauchy's Theorem and Applications [SS]

Goursat's theorem

• **Thm.-** (Goursat) If Ω is an open set in $\mathbb{C}$, $T \subseteq \Omega$ is a triangle whose interior is also contained in Ω then

$$\int_T f(z)dz = 0$$

whenever $f(z)$ is holomorphic in Ω. **Remark.-** In fact, the proof only requires that $f'(z)$ exists on Ω – i.e. no continuity required.

• **Cor.-** Same for any contour that can be bisected into triangles – e.g. rectangle.

Local existence of primitives and Cauchy's theorem on a disk

• **Thm.-** A holomorphic function on an open disk has a primitive on the disk.

• **Thm.-** (Cauchy, on a disk) If f is holomorphic in a disk then

$$\int_\gamma f(z)dz = 0$$

for any closed curve γ on the disk.

Cauchy's integral formulas

• **Thm.-** Suppose f is holomorphic in an open set containing a disk D and its boundary C, where C has positive (i.e. counterclockwise) orientation. Then

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

Proof idea.- Keyhole contour.

• **Remark.-** The same proof applies to any contour that admits a "keyhole"-ification. Observe the integral is zero for any z outside of the contour.

• **Cor.-** If $f(z)$ is holomorphic in Ω then it has infinitely many derivatives in Ω. Moreover, if Ω contains a disk D and its boundary C then for all z in the interior of D

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}d\zeta.$$

• **Cor.-** (Cauchy inequalities) If f is holomorphic in a neighbourhood of the closure of a disk D with boundary C centered at z_0 with radius R then

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{R^n}$$

where $\|f\|_C$ denotes the supremum of f on the circle C.

• **Cor.-** (Liouville's theorem) If f is entire and bounded then f is constant. **Proof idea.-** Show $f' = 0$.

• **Cor.-** (Fundamental Theorem of Algebra) Every non-constant polynomial has a zero in $\mathbb{C}$.

• **Thm.-** Suppose f is holomorphic in a neighbourhood of the closure of a disk D centered at z_0. Then f admits a power series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D$ and the coefficients are given by

$$a_n = \frac{1}{n!} f^{(n)}(z_0).$$

• **Cor.-** The zeros of a non-constant holomorphic function $f(z)$ on a domain are isolated.

• **Cor.-** If f is holomorphic on a domain Ω and its zeros accumulate in Ω then $f = 0$. If $f(z), g(z)$ are holomorphic on Ω and the points where they agree accumulate in Ω then $f = g$.

Further applications

• **Thm.-** (Morera) Suppose f is a continuous function in the open disk D such that for all triangles T contained in D

$$\int_T f(z)dz = 0.$$

Then f is holomorphic. **Proof idea.-** The function f has a holomorphic primitive.

• **Cor.-** If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on Ω that converge uniformly to a function f on compacts then f is holomorphic.

• **Thm.-** Under the hypothesis of the previous corollary, $\{f_n'\}$ converges to f' uniformly on compacts.

• **Thm.-** Let $F(z, s)$ be a continuous function on $\Omega \times [0, 1]$ where $\Omega \subseteq \mathbb{C}$ is open, and suppose that $F(z, s_0)$ is holomorphic for every $s_0 \in [0, 1]$. Then

$$f(z) := \int_0^1 F(z, s)ds$$

is holomorphic.

• **Thm.-** (Symmetry principle) Let Ω be an open subset of $\mathbb{C}$ that is symmetric with respect to the real line, let Ω^+ be the part of Ω lying (strictly) in the upper half plane, Ω^- be the part lying (strictly) in the lower half plane and $I = \Omega \cap \mathbb{R}$. Suppose f^+ (resp. f^-) is holomorphic in Ω^+ (resp. Ω^-) and that it extends continuously to I. Suppose f^+ and f^- agree on I. Then the function

$$f(z) = \begin{cases} f^+(z) \text{ if } z \in \Omega^+ \\ f^+(z) = f^-(z) \text{ if } z \in I \\ f^-(z) \text{ if } z \in \Omega^- \end{cases}$$

is holomorphic on Ω.

• **Cor.-** (Schwarz's reflection principle) Suppose f is holomorphic in Ω^+ and that it extends continuously onto I, on which it is real valued. Then f can be extended to a holomorphic function F on Ω, where $F(z) = \overline{f(\overline{z})}$ for $z \in \Omega^-$.

• **Thm.-** (Runge's approximation theorem) Any function holomorphic on a neighbourhood of a compact set K can be approximated uniformly on K by rational functions whose singularities are in K^c. If K^c is connected, any function holomorphic in a neighbourhood of K can be approximated uniformly on K by polynomials.

§3: Meromorphic Functions and the Logarithm [SS]

Zeros and poles

• **Def.-** A **point singularity** of a function f is a point z_0 such that f is defined on a deleted neighbourhood of z_0, but not at z_0. A point z_0 is called a **zero** of f if $f(z_0) = 0$.

• **Thm.-** Suppose f is a holomorphic function on Ω, and that $z_0 \in \Omega$ is a zero of f. Then there exists a unique integer n and a holomorphic function g on Ω – with $g(z_0) \neq 0$ – such that $f(z) = (z - z_0)^n g(z)$.

• **Def.-** In the theorem above, n is called the **multiplicity** of f at z_0.

• **Def.-** We say f has a pole at z_0 if it is defined in a deleted neighbourhood of z_0 and $1/f$, defined to be zero at z_0, is holomorphic on a full neighborhood of z_0.

• **Thm.-** If z_0 is a pole of f then there is a unique integer n and a holomorphic function $h(z)$ defined on a neighbourhood of z_0, with $h(z_0) \neq 0$, such that $f(z) = (z - z_0)^{-n} h(z)$ on a neighbourhood of z_0.

• **Def.-** From the above theorem, n is called the **order** of the pole z_0.

• **Thm.-** If $f(z)$ has a pole of order n at z_0 then, on a neighbourhood of z_0,

$$f(z) = a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1} + G(z)$$

where $G(z)$ is holomorphic on a neighbourhood of z_0.

• **Def.-** In the above theorem, $a_{-n}(z - z_0)^{-n} + \cdots + a_{-1}(z - z_0)^{-1}$ is called the **principal part** of $f(z)$ at z_0. The coefficient a_{-1} is called the **residue** of f at z_0, denoted $\operatorname{res}_{z_0} f = a_{-1}$.

• **Thm.-** If f has a pole of order n at z_0 then

$$\operatorname{res}_{z_0} f = \frac{1}{(n-1)!} \lim_{z \to z_0} \left(\frac{d}{dz} \right)^{n-1} (z - z_0)^n f(z).$$

The residue formula

• **Thm.-** Suppose f is holomorphic in an open set containing a circle C and its interior, except for a pole z_0 inside of C. Then

$$\int_C f(z)dz = 2\pi i \operatorname{res}_{z_0} f.$$

• **Cor.-** (Residue formula) Suppose f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at $z_1, \ldots, z_N$ inside γ. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^{N} \operatorname{res}_{z_i} f.$$

Singularities and meromorphic functions

• **Thm.-** (Riemann's theorem on removable singularities) Suppose f is holomorphic on Ω except at a point z_0 in Ω. If f is bounded in $\Omega \setminus \{z_0\}$ then z_0 is a removable singularity of f. **Proof idea.-** By using a keyhole, can show Cauchy's formula still works, and this extends holomorphically onto z_0. **Cor.-** Suppose f has an isolated singularity at the point z_0. Then z_0 is a pole of f if and only if $|f(z)| \to \infty$ as $z \to z_0$.

• **Thm.-** (Casorati-Weierstrass) Suppose f is holomorphic in the punctured disc $D \setminus \{z_0\}$ and that f has an essential singularity at z_0. Then $f(D \setminus \{z_0\})$ is dense on the complex plane. (c.f. Picard's theorem for a stronger result).

• **Def.-** A function f is **meromorphic** in Ω if it is holomorphic in $\Omega \setminus \{z_i\}$ and has at most poles at the $\{z_i\}$. **Remark.-** The $\{z_i\}$ must be isolated and, in particular, they form a countable collection.

• **Def.-** Suppose f is holomorphic for all $|z| > R$ where $R \gg 0$. We say that f has a **pole at infinity** if $F(z) = f(1/z)$ has a pole at $z = 0$. Similarly, f has a **removable singularity** (resp. **essential singularity**) **at infinity** if $F(z)$ has a removable (resp. essential) singularity at $z = 0$. A meromorphic function on $\mathbb{C}$ that is holomorphic at infinity, or has a pole at infinity, is said to be **meromorphic in the extended complex plane**.

• **Thm.-** The meromorphic functions in the extended complex plane are the rational functions. Rational functions are determined up to a constant by the location and multiplicity of the zeros and poles. **Remark.-** We really need the function to be meromorphic on $\mathbb{C}$ to start with – consider $\exp(1/z)$.

Argument principle and applications

• **Thm.-** (Argument principle) Suppose f is meromorphic in an open set containing a circle C and its interior. If f has no poles and never vanishes on C then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = \#\{ \text{ zeros of } f \text{ inside } C \ \} - \#\{ \text{ poles of } f \text{ inside } C \ \}$$

where the zeros and poles are counted with multiplicity. **Proof idea.-** $f'(z)/f(z)$ has at most simple poles. Analyze the residues.

• **Thm.-** (Rouché's theorem) Suppose f and g are holomorphic in an open set containing a circle C and its interior. If

$$|f(z)| > |g(z)| \quad \text{ for all } z \in C$$

then f and $f + g$ have the same number of zeros inside of C.

• **Thm.-** (Open mapping theorem) Non-constant holomorphic functions are open.

• **Thm.-** (Maximum modulus principle) Non-constant holomorphic functions on a domain Ω cannot attain a maximum in Ω. **Cor.-** If f is holomorphic in a *bounded* domain Ω and it extends continuously onto $\partial\Omega$ then f attains its maximum in $\partial\Omega$.

• **Thm.-** (Strict maximum principle) Suppose $f(z)$ is a holomorphic function on *any* domain Ω with $|f(z)| \leq M$ for all $z \in \Omega$. If $|f(z_0)| = M$ for some $z_0 \in \Omega$ then $f(z)$ is constant on Ω. **Remark.-** This version does not require $f(z)$ to extend continuously onto the boundary. This is in [G].

Homotopies and simply connected domains

• **Thm.-** If f is holomorphic in Ω and $\gamma_0 \simeq \gamma_1$ (i.e. γ_0 and γ_1 are homotopic paths in Ω) then

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz.$$

• **Thm.-** Any holomorphic function in a simply-connected domain has a primitive. **Cor.-** If f is holomorphic in a simply-connected domain Ω then $\int_\gamma f(z)dz = 0$ for a closed loop γ.

The complex logarithm

• **Remark.-** Let Ω be a simply connected domain with $1 \in \Omega$ and $0 \notin \Omega$. Then the function $1/z$ has a primitive $\log_\Omega(z)$ in Ω – called a **branch** of the logarithm – satisfying:
(i) $e^{\log_\Omega(z)} = z$ for all $z \in \Omega$.
(ii) $F(r) = \log r$ whenever r is a real number near 1.
We can do just fine without $1 \in \Omega$ as long as we pick our constant carefully.

• **Remark.-** This allows us to define power functions z^α where $\alpha \in \mathbb{C}$ for simply connected domains that don't contain 0.

• **Thm.-** Let $f(z)$ be a nowhere vanishing function holomorphic in a simply connected domain Ω. Then there exists a function $g(z)$ on Ω such that

$$f(z) = e^{g(z)}.$$

Proof idea.- Take $g(z) = \int_\gamma f'(z)/f(z)dz + c_0$.

Hurwitz's Theorem [G]

• **Def.-** A sequence $\{f_k(z)\}$ of holomorphic functions on a domain Ω is said to **converge normally** to $f(z)$ if $\{f_k(z)\}$ converges uniformly on each closed disk contained in Ω – or, equivalently, on every compact set contained in Ω. Also equivalently, if around every point in Ω there is a neighbourhood on which the convergence is uniform.

• **Thm.-** (Hurwitz) Suppose $\{f_k(z)\}$ is a sequence of analytic functions that converges normally to $f(z)$ on a domain Ω, and that $f(z)$ has a zero of order n at $z_0 \in \Omega$. Then there exists a $\rho > 0$ such that for $k \gg 0$ the function $f_k(z)$ has exactly n zeros inside the disk $\{|z - z_0| < \rho\}$, counting multiplicities.

• **Def.-** A holomorphic function on Ω is **univalent** if it is one-to-one – i.e. if it is conformal onto some other domain.

• **Cor.-** If a sequence $\{f_k(z)\}$ of univalent functions converges normally to $f(z)$ then $f(z)$ is either univalent or constant.

§4: The Schwarz Lemma [G]

The Schwarz Lemma

• **Thm.-** (Schwarz Lemma) Let $f(z)$ be analytic for $|z| < 1$ and suppose that $|f(z)| \leq 1$ for all $|z| < 1$ and $f(0) = 0$. Then $|f(z)| \leq |z|$ for $|z| < 1$. Furthermore, if equality holds at some point $z_0 \neq 0$ then $f(z) = \lambda z$ for some $|\lambda| = 1$. **Proof idea.-** Write $f(z) = zg(z)$ and apply maximum principle to $g(z)$ for $|z| < r$ where $0 < r < 1$.

• **Cor.-** If $f(z)$ is analytic for $|z - z_0| < r$ and $|f(z)| \leq M$ for $|z - z_0| < r$ then $|f(z)| \leq M/r|z - z_0|$, where equality holds if and only if $f(z)$ is a multiple of $z - z_0$.

• **Cor.-** Let $f(z)$ be analytic for $|z| < 1$. If $|f(z)| \leq 1$ for $|z| < 1$ and $f(0) = 0$ then $|f'(0)| \leq 1$ with equality if and only if $f(z) = \lambda z$ for some $|\lambda| = 1$.

Conformal Self-Maps of the Unit Disk

• **Lemma.-** If $g(z)$ is a conformal self-map of the (open) unit disk $\mathbb{D}$ with $g(0) = 0$ then $g(z) = e^{i\phi}z$.

• **Thm.-** The conformal self-maps of the unit disk $\mathbb{D}$ are of the form

$$f(z) = e^{i\phi}\frac{z-a}{1-\bar{a}z}$$

where $0 \le \phi < 2\pi$ and $a \in \mathbb{D}$. Moreover, ϕ and a give a one-to-one correspondence between conformal self-maps of $\mathbb{D}$ and $\mathbb{D} \times \partial\mathbb{D}$ – where $a = f^{-1}(0)$, $\phi = \arg f'(0)$.

• **Thm.-** (Pick's lemma) If $f(z)$ is analytic and $|f(z)| < 1$ for $|z| < 1$ then

$$|f'(z)| \le \frac{1-|f(z)|^2}{1-|z|^2} \quad |z| < 1.$$

Proof idea.- Use a conformal self-map so that, after composition, we map 0 to 0 and then use Schwarz's Lemma.

• **Def.-** A **finite Blaschke product** is a rational function of the form

$$B(z) = e^{i\phi}\left(\frac{z-a_1}{1-\bar{a_1}z}\right)\cdots\left(\frac{z-a_n}{1-\bar{a_n}z}\right)$$

where the $a_i \in \mathbb{D}$ and $0 \le \phi \le 2\pi$.

• **Thm.-** If $f(z)$ is continuous for $|z| \le 1$ and analytic for $|z| < 1$ and $|f(z)| = 1$ for $|z| = 1$ then $f(z)$ is a finite Blaschke product. **Proof idea.-** Consider $B(z)$, the finite Blaschke product that has the same zeros – with same multiplicities – as $f(z)$. Then $B(z)/f(z)$ and $f(z)/B(z)$ extend holomorphically $\mathbb{D} \to \mathbb{D}$ with modulus 1 on the boundary.

§5: Conformal Mappings [G]

• **Remark.-** The Möbius transformation $z \mapsto (z-i)/(z+i)$ maps the upper half-plane $\mathbb{H}$ conformally onto the unit disk $\mathbb{D}$.

• A **sector** can be mapped onto $\mathbb{H}$ by the help of a power function, and from there to the unit disk if necessary.

• A **strip** can be rotated to be a horizontal strip. Then e^z maps horizontal strips to sectors.

• A **lunar domain** is a domain whose boundary consists of two circles (or line) segments. If z_0, z_1 are the points of intersection, map z_0 to 0 and z_1 to ∞ using a Möbius transformation. We then get a sector.

§6: Compact families of meromorphic functions [G]

Arzelà-Ascoli Theorem

• **Def.-** Let $E \subseteq \mathbb{C}$ be a subset and $\mathcal{F}$ be a family of functions on E. We say $\mathcal{F}$ is **equicontinuous** at $z_0 \in E$ if for all $\epsilon > 0$ there exists some $\delta > 0$ such that whenever $|z - z_0| < \delta$ then for all $f \in \mathcal{F}$ $|f(z) - f(z_0)| < \epsilon$. We say $\mathcal{F}$ is **uniformly bounded** on E if there is some $M > 0$ such that $|f(z)| \leq M$ for all $z \in E$ and $f \in \mathcal{F}$.

• **Thm.-** (Arzelà-Ascoli - $\mathbb{C}$-version) Let $\Omega \subseteq \mathbb{C}$ be a domain and $\mathcal{F}$ be a family of *continuous* functions on Ω that is uniformly bounded on compacts. Then the following are equivalent:
(i) $\mathcal{F}$ is equicontinuous on Ω.
(ii) $\mathcal{F}$ is **normally sequentially compact**, i.e. every sequence in $\mathcal{F}$ has a subsequence that converges normally.

• **Thm.-** (Arzelà-Ascoli - $\hat{\mathbb{C}}$-version) Let $\Omega \subseteq \mathbb{C}$ be a domain and $\mathcal{F}$ be a family of *continuous* functions from D to $\hat{\mathbb{C}}$. Then the following are equivalent:
(i) $\mathcal{F}$ is equicontinuous on Ω.
(ii) $\mathcal{F}$ is normally sequentially compact.

• **Remark.-** In the last theorem we use the spherical metric on $\hat{\mathbb{C}}$. Observe that no boundedness assumptions are needed on the $\hat{\mathbb{C}}$-version – I suspect because there is a general version where we only need the target to be compact.

Compactness of families of functions

• **Lemma.-** If $\mathcal{F}$ is a family of analytic functions on a domain Ω such that $\mathcal{F}'$, the family of derivatives of functions in $\mathcal{F}$, is uniformly bounded then $\mathcal{F}$ is equicontinuous at every point in E.

• **Thm.-** (Montel – weak version) Suppose $\mathcal{F}$ is a family of analytic functions on a domain Ω that is *uniformly bounded on compacts*. Then every sequence in $\mathcal{F}$ has a normally convergent subsequence. **Proof idea.-** Using Cauchy estimates we show $\mathcal{F}'$ is uniformly bounded on compacts, thus $\mathcal{F}$ is equicontinuous. Then use Arzelà-Ascoli to obtain a subsequence that converges – but this sequence may depend on the compact, so we need to use a diagonalization argument.

• **Sample application:** Fix a domain Ω and a point $z_0 \in \Omega$. We consider the family $\mathcal{F}$ of analytic functions f on Ω with $|f(z)| \leq 1$ for all $z \in \Omega$. Then the supremum $\sup\{|f'(z_0)| : f \in \mathcal{F}\}$ is attained. (c.f. Ahlfors function).

Marty's Theorem

• We extend the notion of normal convergence to meromorphic functions by using the spherical metric on $\hat{\mathbb{C}}$.

• **Thm.-** If a sequence $\{f_n(z)\}$ of meromorphic functions converges normally to $f(z)$ on a domain Ω then $f(z)$ is either meromorphic or $f(z) \equiv \infty$. If the initial $\{f_n(z)\}$ were analytic then either $f(z)$ is analytic or $f(z) \equiv \infty$.

• **Def.-** A family $\mathcal{F}$ of meromorphic functions on Ω is said to be a **normal family** if every sequence in $\mathcal{F}$ has a subsequence that converges normally in Ω.

• **Def.-** Given a meromorphic function f, regarded as a map $\Omega \to \hat{\mathbb{C}}$, its **spherical derivative** at the point z is

$$f^{\#}(z) := \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

• **Lemma.-** If $f_k \to f$ normally on Ω then $f_k^\# \to f^\#$ normally on Ω.

• **Thm.-** (Marty) A family $\mathcal{F}$ of meromorphic functions on Ω is normal if and only if the family of spherical derivatives is bounded uniformly on compacts.

Strong Montel and Picard

• **Thm.-** (Zalcman's Lemma) Suppose $\mathcal{F}$ is a family of meromorphic functions on a domain Ω that is *not* normal. Then there exist points $z_n \in \Omega$ with $z_n \to z \in \Omega$, $\rho_n > 0$ with $\rho_n \to 0$ and functions $f_n \in \mathcal{F}$ such that $g_n(\zeta) := f_n(z_n + \rho_n\zeta)$ converges normally to a meromorphic function $g(\zeta)$ on $\mathbb{C}$ with $g^\#(0) = 1$ and $g^\#(\zeta) \leq 1$ for $\zeta \in \mathbb{C}$.

• **Def.-** A family $\mathcal{F}$ of meromorphic functions on Ω **omits** a value $w_0 \in \hat{\mathbb{C}}$ if $w_0 \notin f(\Omega)$ for all $f \in \mathcal{F}$.

• **Thm.-** (Montel – strong) A family $\mathcal{F}$ of meromorphic functions on a domain Ω that omits three values of $\hat{\mathbb{C}}$ is normal.

• **Def.-** Suppose f is meromorphic on a punctured neighbourhood of z_0. A value $w_0 \in \hat{\mathbb{C}}$ is an **omitted value at** z_0 if there exists some $\delta > 0$ such that $f(z) \neq w_0$ for all $0 < |z - z_0| < \delta$. Thus w_0 is *not* an omitted value if there is a sequence $z_n \to z_0$ with $f(z_n) = w_0$.

• **Thm.-** (Picard's big theorem) Suppose $f(z)$ is meromorphic on a punctured neighborhood of z_0. If $f(z)$ omits three values at z_0 then $f(z)$ extends to be meromorphic at z_0 – i.e. z_0 is a pole or removable.

• **Thm.-** (Picard's little theorem) A nonconstant entire function assumes every value in the complex plane with at most one exception.

References

[SS]: Stein and Shakarchi.

[G]: Gamelin.

[F]: Folland.