Basic Exam, Fall 2013

Instructions: Write your UCLA student number on each page of your solutions. Do not write your name. Work 10 of the 12 problems, at least 4 of the first 6 and at least 4 of the last 6, and indicate here which 10 problems you want to have graded: (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12). Each problem is worth 10 points, but different parts of a problem may have different values.

1. When $\{a_n\}$ is a sequence of positive real numbers, $a_n > 0$, define $P_n = \Pi_{j=1}^n (1+a_j)$. Prove that $\lim_{n\to\infty} P_n$ exists and is a *non-zero* real number if and only if $\sum_n a_n < \infty$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a nondecreasing real-valued function, i.e. if $x < y$ then $f(x) \le f(y)$.

(a) Prove that $\{x \in \mathbb{R} : f \text{ is not continuous at } x\}$ is countable.

(b) Let $S \subset \mathbb{R}$ be a countable set. Prove there exists nondecreasing $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, f is not continuous at x if and only if $x \in S$.

3. Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous one-to-one function. By definition the length of the range $\gamma([0, 1])$ is

$$L(\gamma) = \sup\Big\{\sum_{j=1}^{n-1} |\gamma(t_{j+1}) - \gamma(t_j)| : 0 \le t_1 < t_2 < \ldots < t_n \le 1,\ n < \infty\Big\}$$

where $|(x, y)| = \sqrt{x^2 + y^2}$ when $(x, y) \in \mathbb{R}^2$.

(a) Suppose $f(t)$ is continuous and nondecreasing on $[0, 1]$, and let $\gamma(t) = (t, f(t))$ (so that the range of γ is the graph of f). Prove

$$L(\gamma) \le 1 + (f(1) - f(0)).$$

(b) Show there exists continuous nondecreasing $f(t)$ on $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$ and such that $L(\gamma) = 2$ when $\gamma(t) = (t, f(t))$,

4. Let $f(x, y)$ be a continuous real-valued function on the plane $\mathbb{R}^2$. Assume that for every square $S = \{a < x < a + \frac{1}{n}, b < y < b + \frac{1}{n}\}$, where a and b are rational numbers and $n = 1, 2, \ldots,$

$$\int\int_S f(x, y)dxdy = 0.$$

Prove $f(x, y) = 0$ for all (x, y).

5. A function $f : \mathbb{R}^d \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \text{ for all } \quad x, y \in \mathbb{R}^d,\ 0 \leq t \leq 1.$$

Assume that f is continuously differentiable such that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0, \quad x, y \in \mathbb{R}^d$$

where ∇f is the gradient of f and $\cdot$ is the inner product on $\mathbb{R}^d$. Prove that f is convex.

6. Let X be a compact metric space, let $\{x_n\}$ be a sequence in X and let $x \in X$. Assume that for every subsequence $\{y_n\}$ of $\{x_n\}$ there is a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\{z_n\}$ converges to x. Prove $\{x_n\}$ converges to x.

7. Let $z_1, z_2, ..., z_n$ be distinct complex numbers and for $1 \leq j \leq n$, let m_j be a non-negative integer. Write $N + 1 = \sum_{j=0}^{n}(1 + m_j)$. Prove that given any array of $N + 1$ complex numbers

$$c_{j,k}, \quad 1 \leq j \leq n, \quad 0 \leq k \leq m_j,$$

there is a unique polynomial $P(z)$ of degree at most N such that for all j, k,

$$P^{(k)}(z_j) = c_{j,k}$$

where $P^{(k)}$ denotes the $k-$th derivative, i.e. $(z^n)^{(2)} = n(n-1)z^{n-2}, n \geq 2$.

8. An orthogonal projection on a finite dimensional inner product space V is an endomorphism P that satisfies $P^2 = P$ and $im(P) = ker(P)^{\perp}$. Suppose $V = \mathbb{R}^3$ and P is an orthogonal projection with diagonal matrix entries $p_{1,1} = 2/3$, $p_{2,2} = 1/2$, $p_{3,3} = 5/6$. Find all matrices that P could be. (Hint: it's a very small finite set!).

9. Let A be an endomorphism of a vector space V of dimension d over a field F. Show, from first principles (i.e. do not use Jordan form or the Cayley-Hamilton theorem) that A satisfies a polynomial $P(X) \in F[X]$ of degree at most d.

10. Let A be an n by n Hermitian matrix and let, for $j \in [1, n]$, A_j be the submatrix consisting of the entries of A in the first j row and columns of A. Suppose that $det(A_j) \neq 0$ and $det(A_1) > 0$. Give and prove a rule in terms of the signs of the $det(A_j)$ to determine the signature of the Hermitian form defined by A.

11. Define a normal linear transformation N on a finite dimensional complex inner product space V. Supposing that N is normal, show that there exists an orthogonal basis of V consisting of eigenvectors of N.

12. Suppose A is an endomorphism of a complex vector space with characteristic polynomial $C_A(X) = X^4 - 6X^3 + 13X^2 - 12X + 4$. How many similarity (i.e. conjugacy) classes of elements can have this characteristic polynomial? Suppose also that the minimal polynomial $M_A(X)$ of A is equal to $C_A(X)$. How many classes satisfy this additional condition? Prove your answers, quoting any general theorems you need.