BASIC EXAM: FALL 2014

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. Do not write your name anywhere on the exam.

Work 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want to have graded.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. Show that the function

$$H(x,y) = x^2 + y^2 + |x-y|^{-1}$$

achieves its global minimum somewhere on the set $\{(x,y) \in \mathbb{R}^2 : x \neq y\}$.

Problem 2. Let A, B be two closed subsets of $\mathbb{R}^n$ such that $A \cup B$ and $A \cap B$ are connected. Prove that A is connected.

Problem 3. Let $f_n : [0,1] \to \mathbb{R}$ be monotonically increasing continuous functions. Assume that f_n converge pointwise to a continuous function $f : [0,1] \to \mathbb{R}$. Prove that f_n converge uniformly to f.

Problem 4. Let $f_n : [-2,2] \to [0,1]$ be a sequence of convex functions. Show that there is a subsequence which converges uniformly on $[-1,1]$.

Problem 5. Consider the following sequence:

$$a_1 = \sqrt{2} \quad \text{and} \quad a_{n+1} = \sqrt{2 + a_n} \quad \text{for all } n \geq 1.$$

Prove that this sequence converges and find its limit.

Problem 6. Let $f : [0,1] \to \mathbb{R}$ be a C^1 function. Prove that

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \left| f\left(\tfrac{k+1}{n}\right) - f\left(\tfrac{k}{n}\right) \right| = \int_0^1 |f'(t)|\, dt.$$

Problem 7. Among all solutions to the system

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 5 & 7 \\ -2 & -1 & 1 & 3 \end{pmatrix} x = \begin{pmatrix} 2 \\ 7 \\ -1 \end{pmatrix},$$

find the solution with minimal length.

Problem 8. Compute the eigenvalues of the following $n \times n$ matrix:

$$M = \begin{pmatrix} k & 1 & 1 & \cdots & 1 \\ 1 & k & 1 & \cdots & 1 \\ 1 & 1 & k & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & k \end{pmatrix}.$$

Use the eigenvalues to compute $\det(M)$.

Problem 9. Let $A \neq 0$ be an $n \times n$ complex matrix. Prove that there is a matrix B, such that B and $A + B$ have no eigenvalues in common.

Problem 10. What is the largest number of 1's an invertible 0-1 matrix of size $n \times n$ can have? You must show both that this number is possible and that no larger number is possible.

Problem 11. Suppose a 4×4 integer matrix has four distinct real eigenvalues $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$. Prove that $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 \in \mathbb{Z}$.

Problem 12. Let $A = \big(a_{ij}\big)_{1 \le i,j \le n}$, where $a_{ij} = 1/(i + j - 1)$. Prove that A is positive definite.