BASIC EXAM: SPRING 2016

Test instructions:

Write your UCLA ID number on the upper right corner of each sheet of paper you use. Do not write your name anywhere on the exam.

Work out 10 problems, including at least 4 of the first 6 problems and at least 4 of the last 6 problems. Clearly indicate which 10 problems you want us to grade.

1	2	3	4
5	6	7	8
9	10	11	12

Problem 1. For $a < b$ real numbers, let $f\colon [a,b] \times [a,b] \to \mathbb{R}$ be such that

(1) for each $y \in [a,b]$, $x \mapsto f(x,y)$ is non-increasing and continuous on $[a,b]$,
(2) for each $x \in [a,b]$, $y \mapsto f(x,y)$ is non-decreasing and continuous on $[a,b]$.

Prove that $g(x) := f(x,x)$ is continuous on $[a,b]$.

Problem 2. For $a < b$ real numbers and $f\colon [a,b] \to \mathbb{R}$ a function, do as follows:

(1) Define what it means for f to be Riemann integrable on $[a,b]$.
(2) Let $\{x_n\}_{n=1}^{\infty} \subset [a,b]$ be a sequence such that $\lim_{n\to\infty} x_n$ exists and suppose that $f\colon [a,b] \to \mathbb{R}$ is defined by

$$f(x) := \begin{cases} 1 & \text{if } x \notin \{x_n\}_{n=1}^{\infty}, \\ 0 & \text{else.} \end{cases}$$

Using your definition, prove that f is Riemann integrable on $[a,b]$.

Problem 3. Suppose $f\colon [0,1] \to \mathbb{R}$ is a continuously differentiable function. Show that the limit

$$\lim_{n\to\infty} n \left(\sum_{k=0}^{n} f\left(\tfrac{k}{n}\right) - n \int_0^1 f(x)\mathrm{d}x \right)$$

exists and compute its value.

Problem 4. Given continuous functions $\alpha\colon [0,1] \to \mathbb{R}$ and $\beta\colon [0,1] \to [0,1)$, define functions $f_n\colon [0,1] \to \mathbb{R}$ by the recursion

$$f_{n+1}(x) = \alpha(x) + \int_0^x \beta(t)f_n(t)\,\mathrm{d}t$$

with $f_0(x) := 0$ for all $x \in [0,1]$. Prove that, for each $x \in [0,1]$, the limit

$$f(x) := \lim_{n\to\infty} f_n(x)$$

exists and compute its value.

Problem 5. Let $f,g\colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable functions such that g attains value zero at at least one point. Suppose that $\nabla g \neq 0$ everywhere on $\mathbb{R}^2$ and assume (x_0, y_0) is a point such that

$$f(x_0, y_0) = \inf\{f(x,y)\colon x,y \in \mathbb{R},\ g(x,y) = 0\}.$$

Show that then there is a $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0).$$

Here ∇f denotes the gradient of f.

Problem 6. A metric ρ in a metric space (X, ρ) is said to be an *ultrametric* if

$$\forall x, y, z \in X: \qquad \rho(x, y) \leq \max\{\rho(x, z), \rho(y, z)\}.$$

Prove that, in this metric, every open ball $\{y \colon \rho(x, y) < r\}$ is closed and every closed ball $\{y \colon \rho(x, y) \leq r\}$ is open.

Problem 7. An orthogonal $n \times n$ matrix A is called *elementary* if the corresponding linear transformation $L_A : \mathbb{R}^n \to \mathbb{R}^n$ fixes an $(n - 2)$-dimensional subspace. Prove that every orthogonal matrix M is a product of at most $(n - 1)$ elementary orthogonal matrices.

Problem 8. Let $A = (a_{ij})$ be a 2×2 real matrix such that

$$a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2 \; < \; \frac{1}{10}\,.$$

Prove that $(I + A)$ is invertible.

Problem 9. Let $v_1 = (0, 1, x)$, $v_2 = (1, x, 1)$, $v_3 = (x, 1, 0)$. Find all $x \in \mathbb{R}$ for which $\{v_1, v_2, v_3\}$ are linearly independent over $\mathbb{R}$. Similarly, find all $x \in \mathbb{R}$ for which $\{v_1, v_2, v_3\}$ are linearly independent over $\mathbb{Q}$.

Problem 10. Let S be a subset of $Mat(3, \mathbb{C})$, the set of 3×3 matrices over $\mathbb{C}$. The set S is called *dense* if every matrix in $Mat(3, \mathbb{C})$ is a limit of a sequence of matrices in S.

   a) Prove that the set of matrices with distinct eigenvalues is dense in $Mat(3, \mathbb{C})$.

   b) Prove that the set of matrices with one Jordan block is not dense in $Mat(3, \mathbb{C})$.

Problem 11. Let M_n be the following 3-diagonal $n \times n$ matrix with $a, b > 0$:

$$M_n = \begin{pmatrix} a & b & 0 & \cdots & 0 & 0 \\ -b & a & b & \cdots & 0 & 0 \\ 0 & -b & a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a & b \\ 0 & 0 & 0 & \cdots & -b & a \end{pmatrix}.$$

Prove that $\det M_n > 0$ for all n. Prove that the limit

$$\lim_{n \to \infty} \log \det M_n$$

exists and compute its value.

Problem 12. Let A be a symmetric $n \times n$ real matrix, $n \geq 4$, and let $v_1, \ldots, v_4 \in \mathbb{R}^n$ be nonzero vectors. Suppose $A v_k = (2k - 1) v_k$ for all $1 \leq k \leq 4$. Prove that $v_1 + 2v_2$ is orthogonal to $3v_3 + 4v_4$.