# BASIC EXAM: FALL 2017

**Test instructions:**

Write your UCLA ID number on the upper right corner of each sheet of paper you use. **Do not write your name anywhere on the exam!!!**

All answers must be justified. If you wish to use a known theorem, make sure to give a full and precise statement.

Work out FIVE of the linear algebra problems (1-6) and FIVE of the analysis problems (7-12). Clearly indicate which 10 problems you want us to grade. To pass the exam successfully, candidates must fare satisfactorily in both parts.

| 1 | 2 | 3 | 4 |
|---|---|---|---|
| | | | |
| 5 | 6 | 7 | 8 |
| | | | |
| 9 | 10 | 11 | 12 |
| | | | |

**Problem 1.** Let $V = \{f(X) = a_0 + a_1X + a_2X^2 + a_3X^3 \mid a_0, \ldots, a_3 \in \mathbb{C}\}$ be the complex vector space of polynomials in the variable $X$, of degree at most 3.

(a) [2 pts] Show that $V$ is an inner product space with $\langle f, g\rangle = \int_{-1}^{1} f(t)\,\overline{g(t)}\,dt$.

(b) [8 pts] Find an orthonormal basis of $V$.

**Problem 2.** Let $n \geq 1$ be an integer and $A$ and $B$ be $n \times n$-matrices.

(a) [5 pts] Show that $AB$ and $BA$ have the same characteristic polynomial if $A$ is invertible.

(b) [5 pts] Is the result true without assuming invertibility? Prove your claim.

**Problem 3.** [10 pts] Solve the following linear system of differential equations, for two functions $x_i : \mathbb{R} \to \mathbb{R}$, for $i = 1$ and 2, with derivatives $x_i'(t) = \frac{dx_i}{dt}(t)$:

$$\begin{cases} x_1' &= 6\,x_1 - x_2 \\ x_2' &= 2\,x_1 + 3\,x_2 \end{cases}$$

**Problem 4.** Let $V$ be a vector space over the field $F = \mathbb{R}$ and let $V^* = \mathrm{Lin}_F(V, F)$ be the dual space (of $F$-linear maps from $V$ to $F$). Let $\mathcal{B} = \{e_i\}_{i\in I}$ be a basis of $V$. For each $i \in I$, define the dual forms $e_i^\# \in V^*$ by the rule

$$e_i^\#(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

(a) [2 pts] Show that the vectors $\{e_i^\#\}_{i\in I}$ are linearly independent in $V^*$.

(b) [8 pts] Give necessary and sufficient conditions on $V$ for these vectors to form a basis of $V^*$. Prove your claim.

**Problem 5.** Let $V$ and $W$ be two infinite-dimensional vector spaces over the field $F = \mathbb{C}$. Let $\mathrm{Lin}_F(V, W)$ be the $F$-vector space of $F$-linear maps from $V$ to $W$.

(a) [4 pts] Is $X = \{f \in \mathrm{Lin}_F(V, W) \mid f \text{ has finite rank}\}$ a subspace of $\mathrm{Lin}_F(V, W)$?

(b) [4 pts] Same question for $Y = \{f \in \mathrm{Lin}_F(V, W) \mid \mathrm{Ker}(f) \text{ is finite dimensional}\}$.

(c) [2 pts] What is the intersection $X \cap Y$?

Prove all claims in full detail.

**Problem 6.** For each of the following three fields $F$ (separately), is it true that every symmetric matrix $A \in \mathrm{M}_{2\times 2}(F)$ is diagonalizable?

(a) [2 pts] For $F = \mathbb{R}$.

(b) [3 pts] For $F = \mathbb{C}$.

(c) [5 pts] For $F = \mathbb{F}_3 = \mathbb{Z}/3\,\mathbb{Z}$, the field with 3 elements.

Supply proofs/counterexamples (or cite the relevant theorems) for all parts of this problem.

**Problem 7.** [10 pts] Let $\{a_n\}_{n=1}^{\infty}$ be a non-increasing sequence of positive real numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. Prove that

$$\lim_{n\to\infty} na_n = 0.$$

**Problem 8.** Let $a < b$ be real numbers and $f\colon [a,b] \to \mathbb{R}$ a function such that $L(x) = \lim_{y\to x} f(y)$ exists for all $x \in [a,b]$ (with one-sided limits at $x = a, b$).

(a) [4 pts] Prove that $L$ is continuous on $[a,b]$.

(b) [3 pts] Prove that $\{x \in [a,b]\colon f(x) \neq L(x)\}$ is countable.

(c) [3 pts] Prove that $f$ is Riemann integrable.

**Problem 9.** [10 pts] Let $(X,\rho)$ be a complete metric space and $f\colon X \to X$ a function. Writing $f^n$ for the $n$-th iterate of $f$, denote

$$c_n := \sup_{\substack{x,y\in X\\ x\neq y}} \frac{\rho\big(f^n(x), f^n(y)\big)}{\rho(x,y)}.$$

Assuming that $\sum_{n=1}^{\infty} c_n < \infty$, prove that $f$ has a unique fixed point in $X$.

**Problem 10.** [10 pts] Let $a < b$ be real numbers and $f\colon [a,b] \to \mathbb{R}$ a continuous function such that $\int_a^b f(x)x^n \mathrm{d}x = 0$ for each integer $n \geq 0$. Prove that $f = 0$.

**Problem 11.** [10 pts] Prove Young's inequality: Let $p, q \in (1,\infty)$ obey $\frac{1}{p} + \frac{1}{q} = 1$. Then for each $a, b \geq 0$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

**Problem 12.** [10 pts] Let $X$ be a compact metric space and $C(X)$ the space of continuous real-valued functions on $X$ endowed with the supremum norm. Let $\mathcal{F} \subset C(X)$ be non-empty. Prove the following version of Arzelà-Ascoli's theorem:

$$\mathcal{F} \text{ is compact} \quad \Leftrightarrow \quad \mathcal{F} \text{ is closed, bounded and equicontinuous}$$

Give precise definitions of all terms used in this equivalence.